An Unhelpful Introduction to Electricity & Magnetism

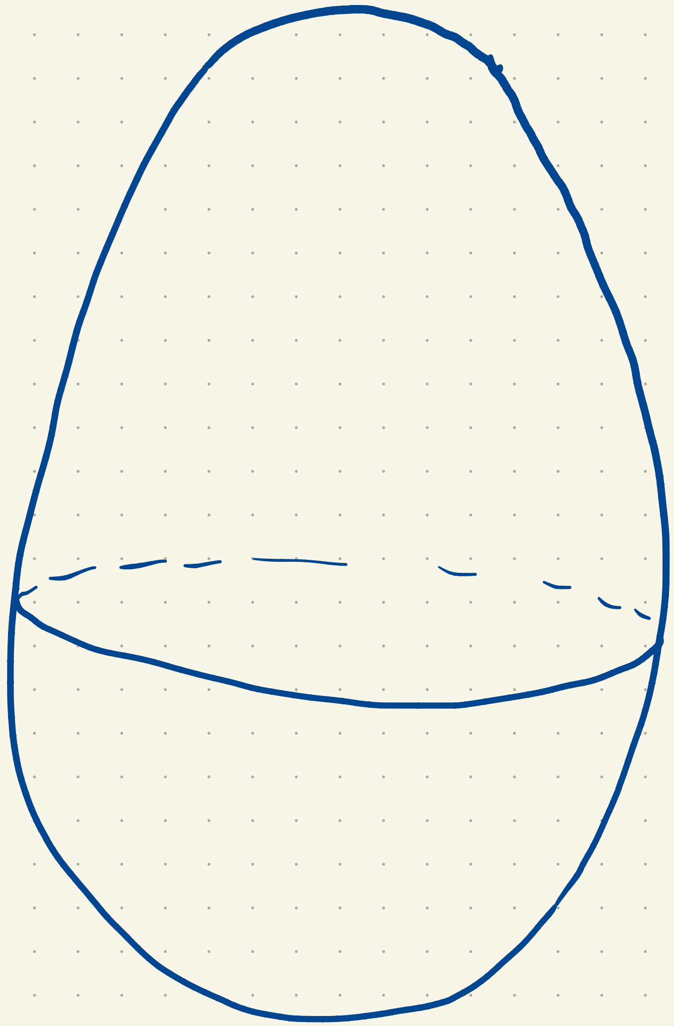
Part III ½: Eggs, Curvature and Two Maxwell Equations

Nov 24, 2020

David Maxwell (damaxwell.github.io)

UAF Mathematics

The Egg



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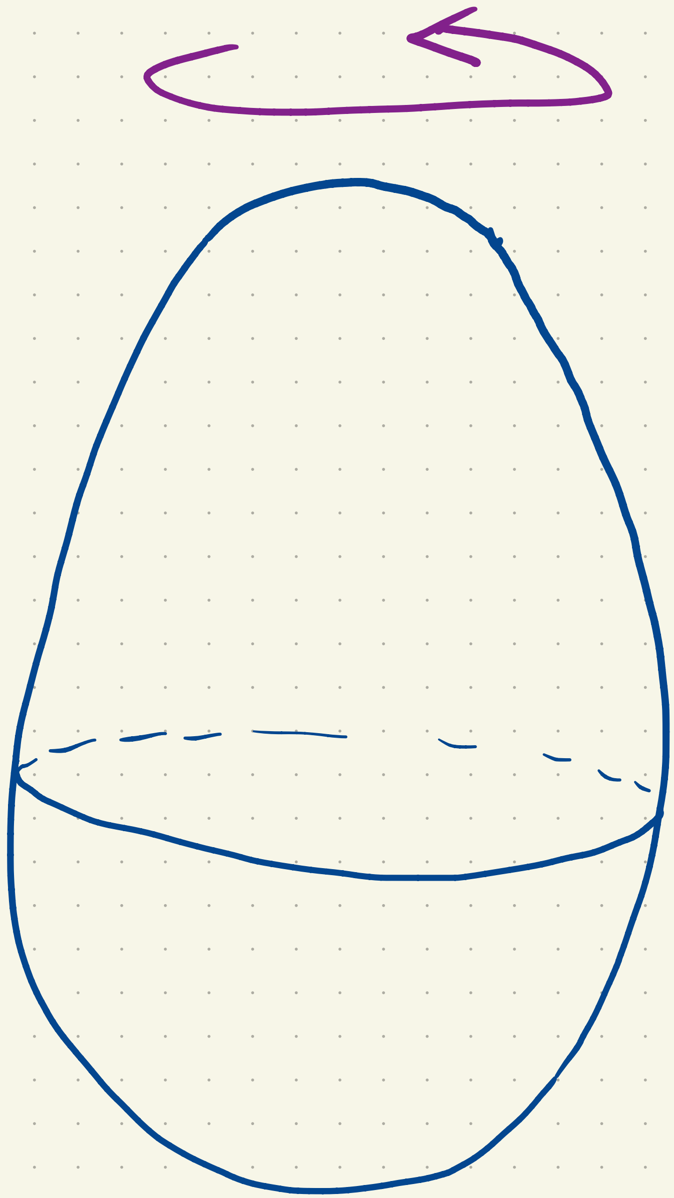

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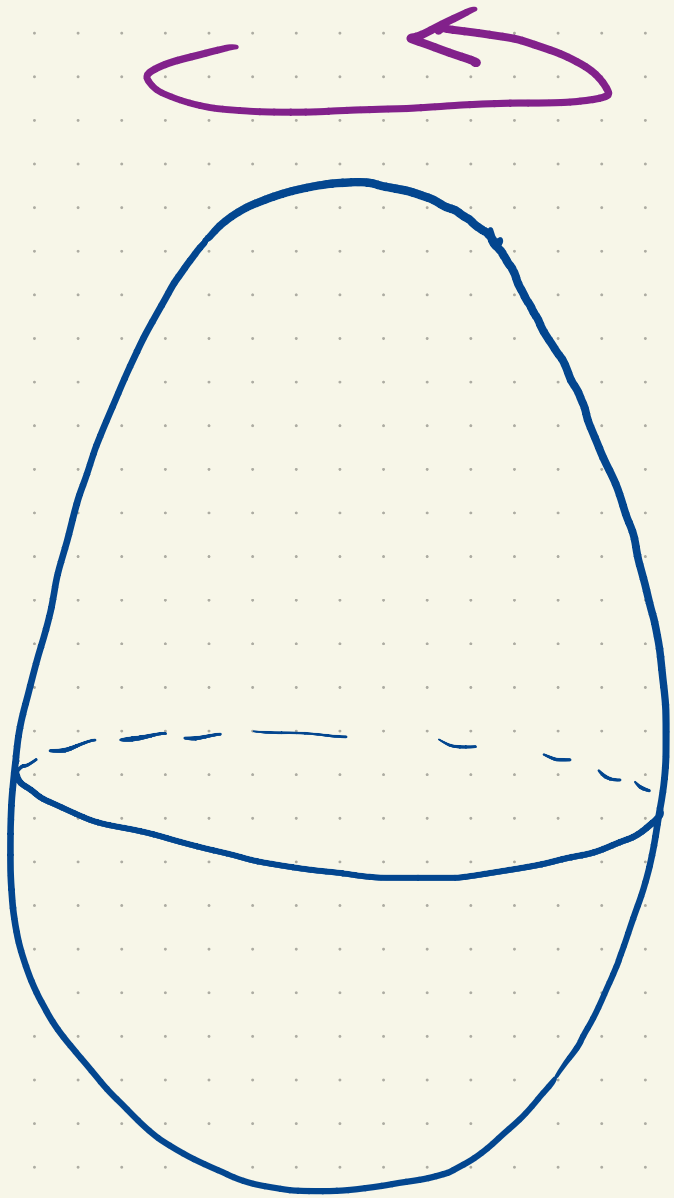
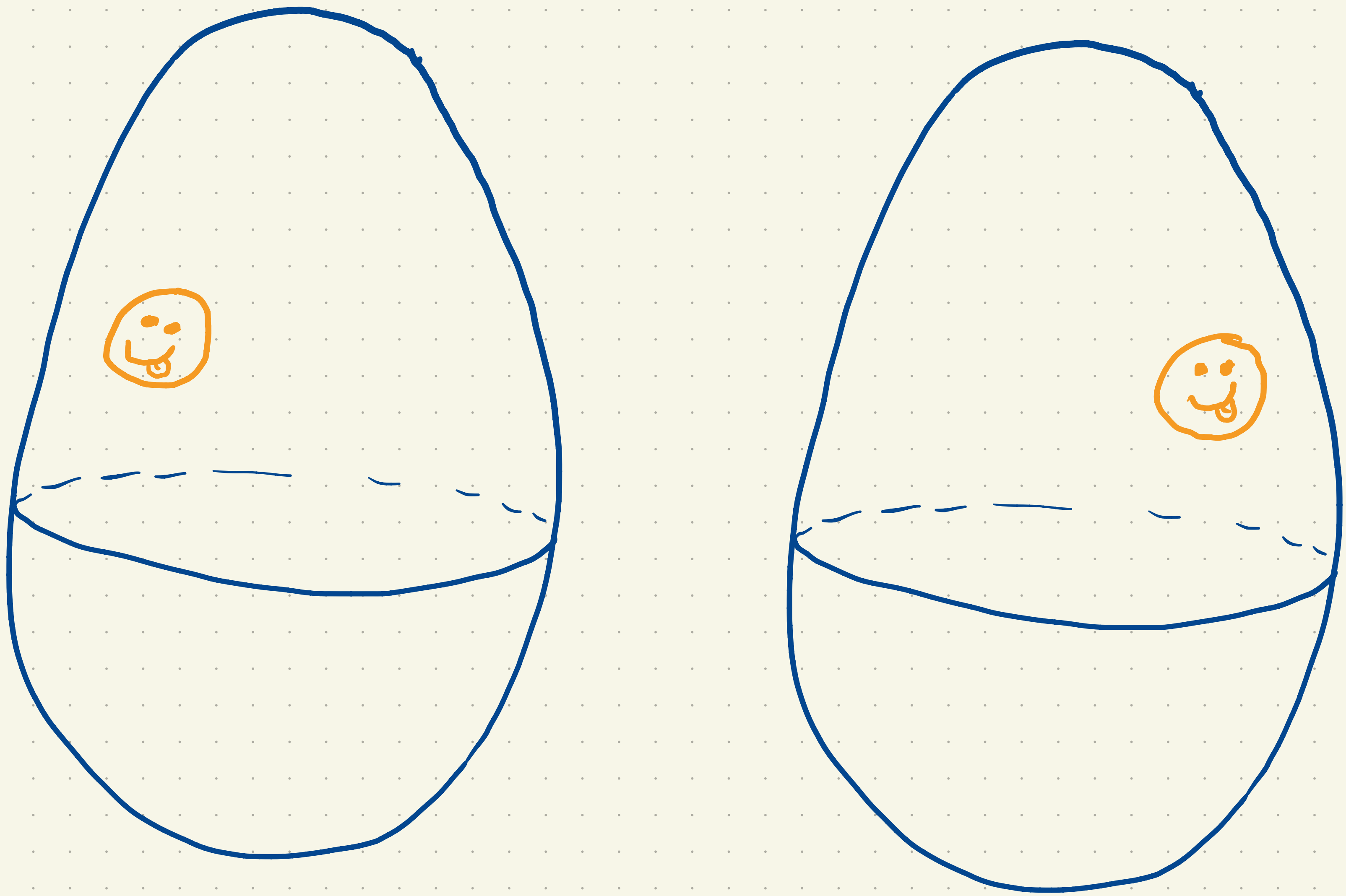


- Symmetry group:

  U(1)

  $\{e^{i\theta} : \theta \in \mathbb{R}\}$

Congruence



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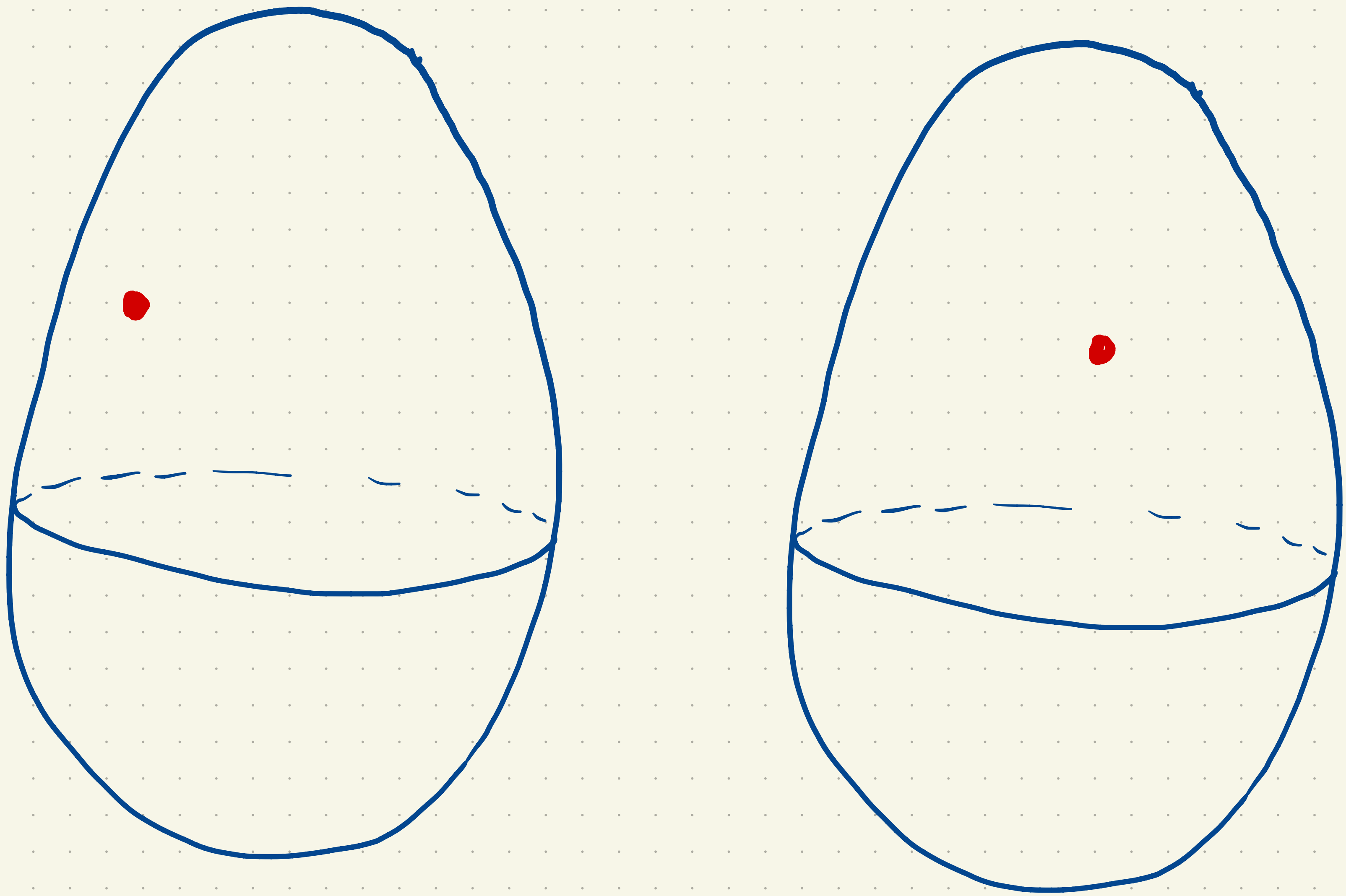
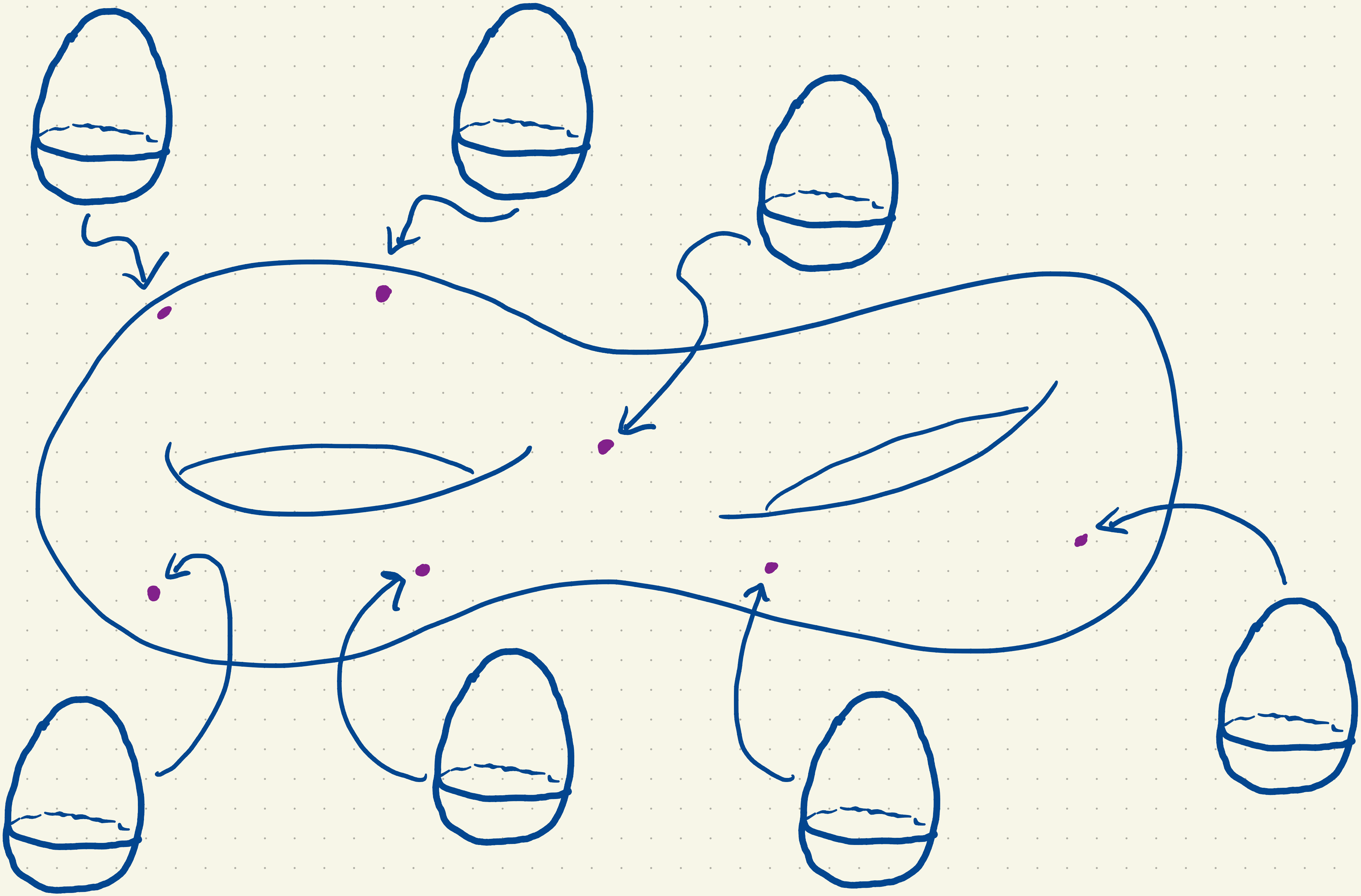


Egg & Toy Space



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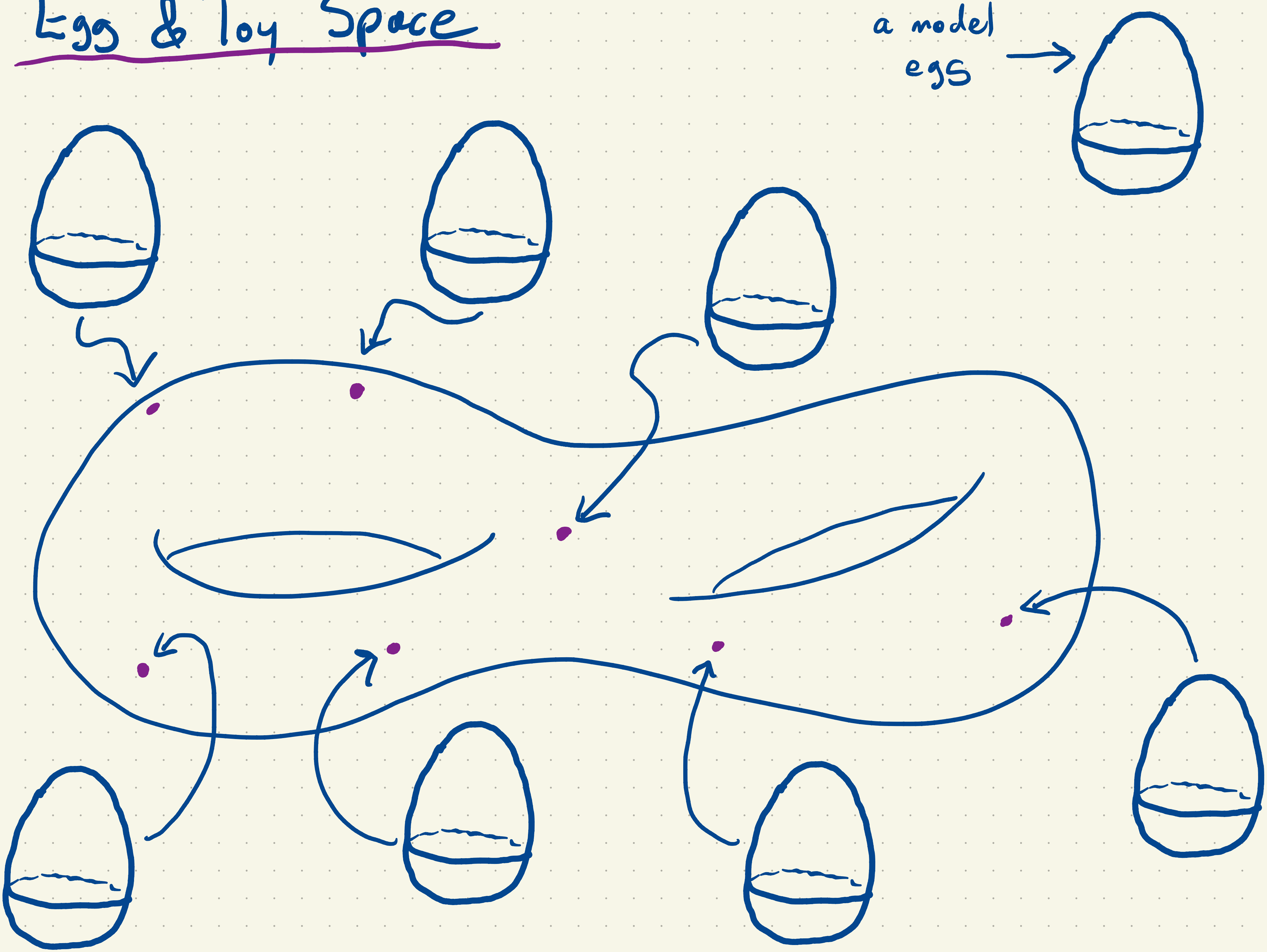

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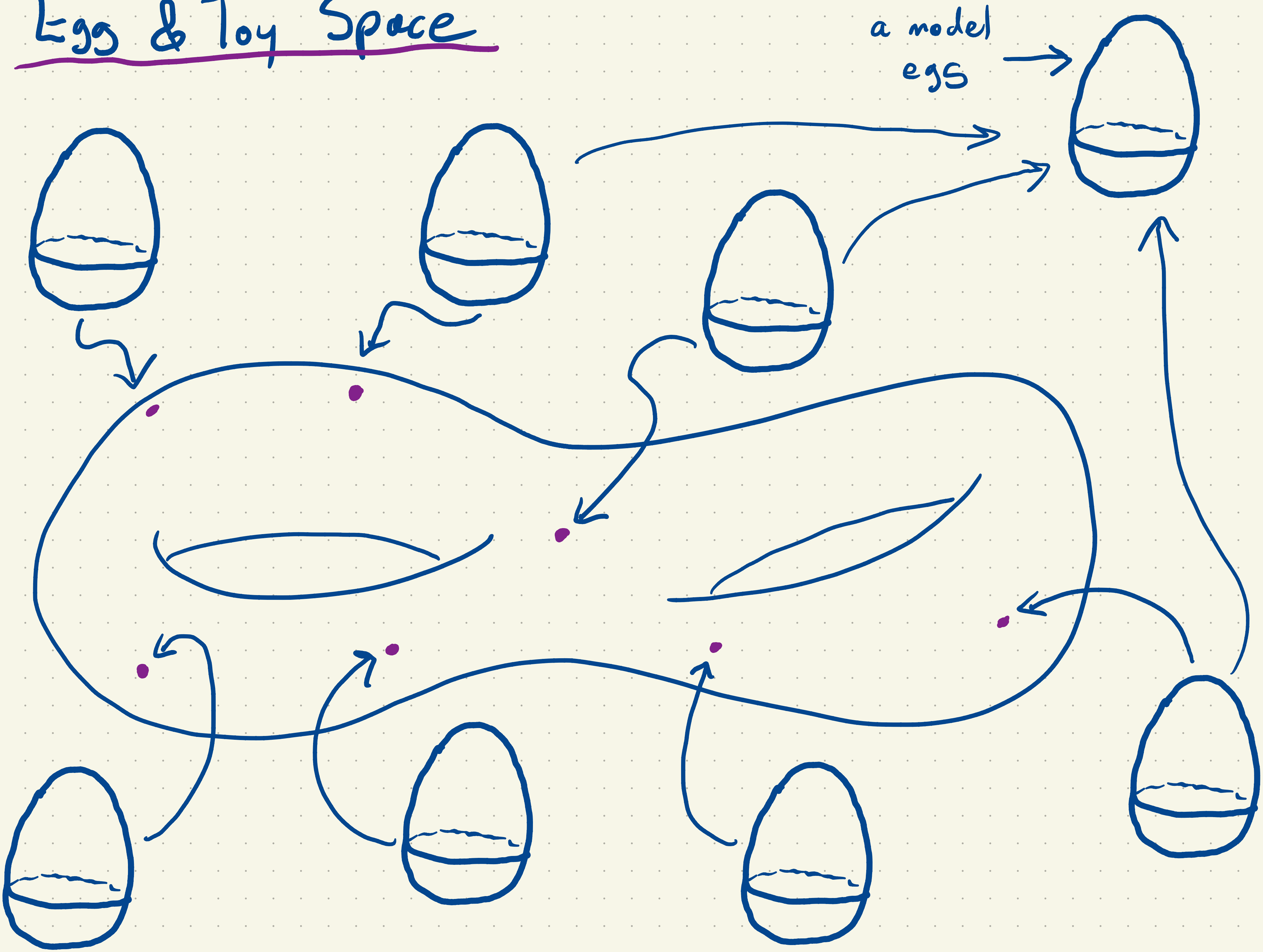

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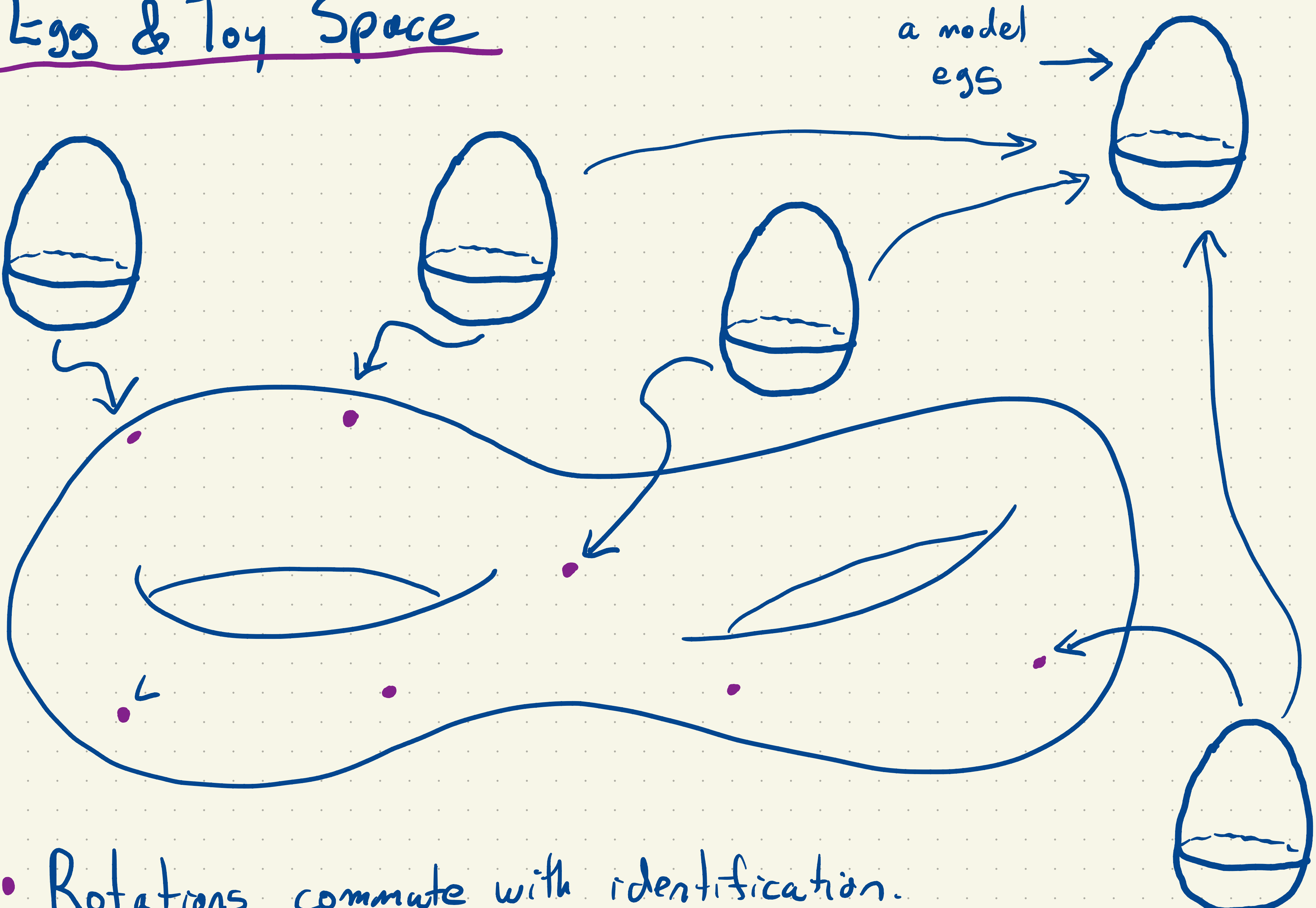
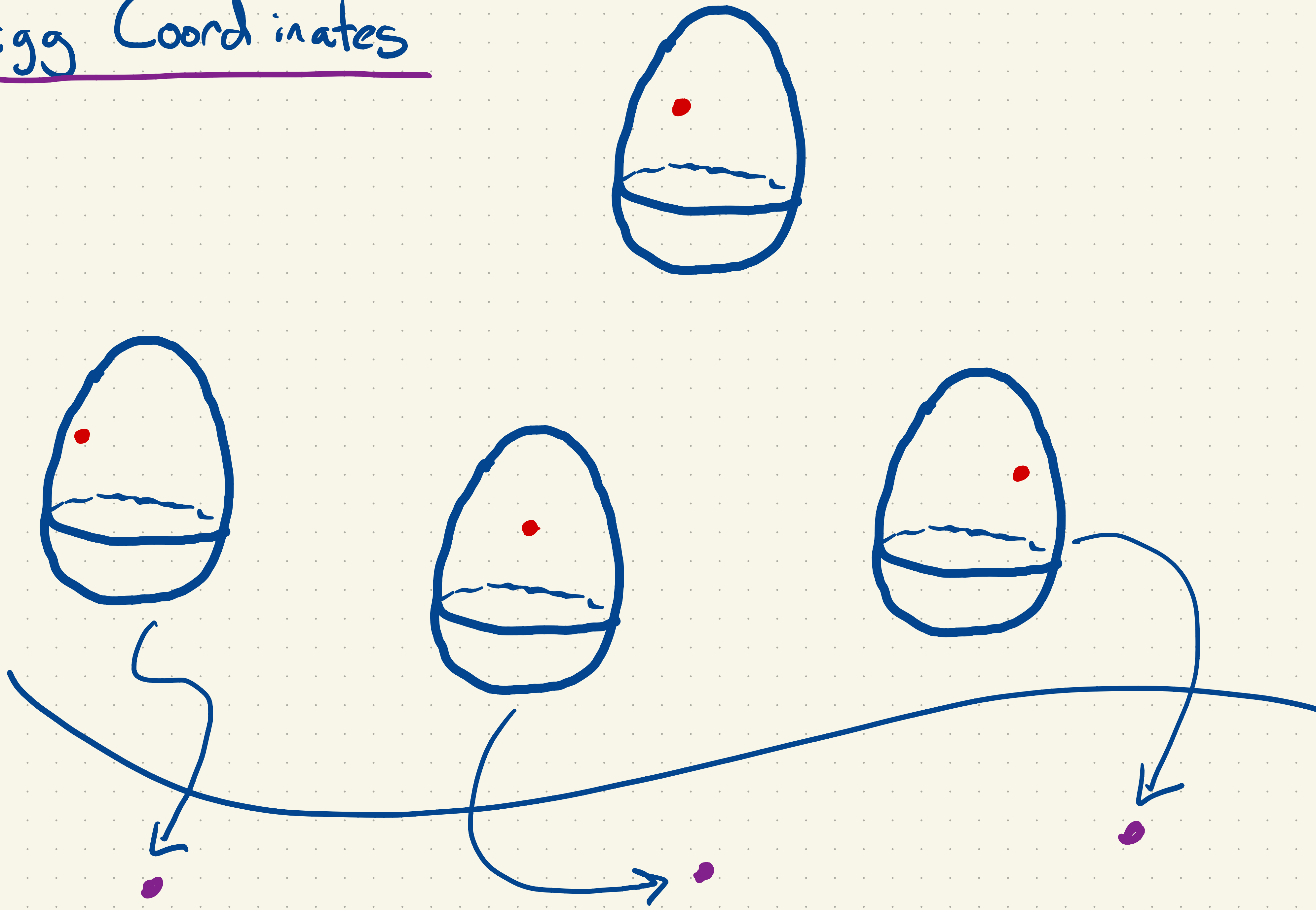


- Rotations commute with identification.
- Any two eggs have a circle's worth of identifications.

Egg Coordinates



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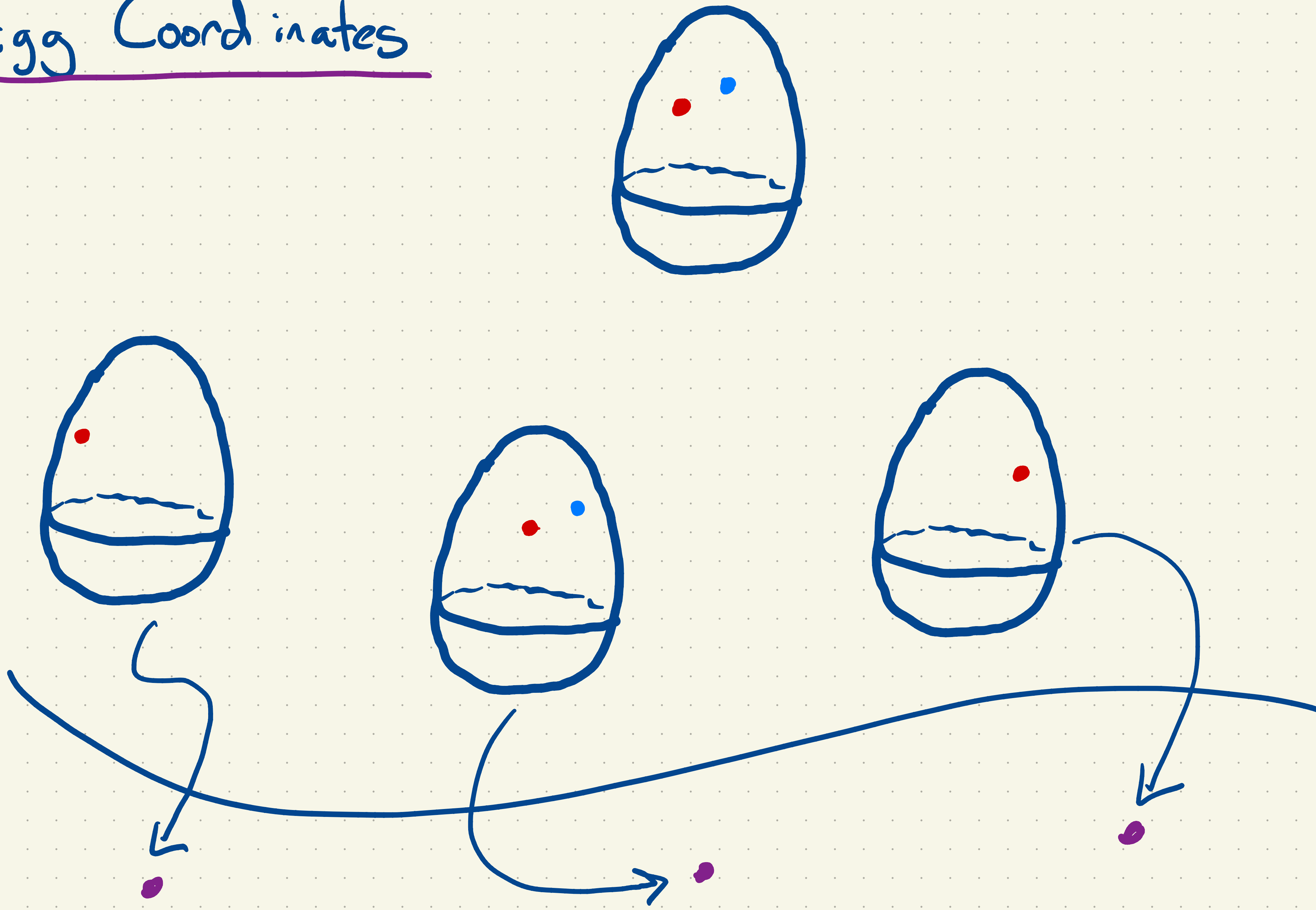

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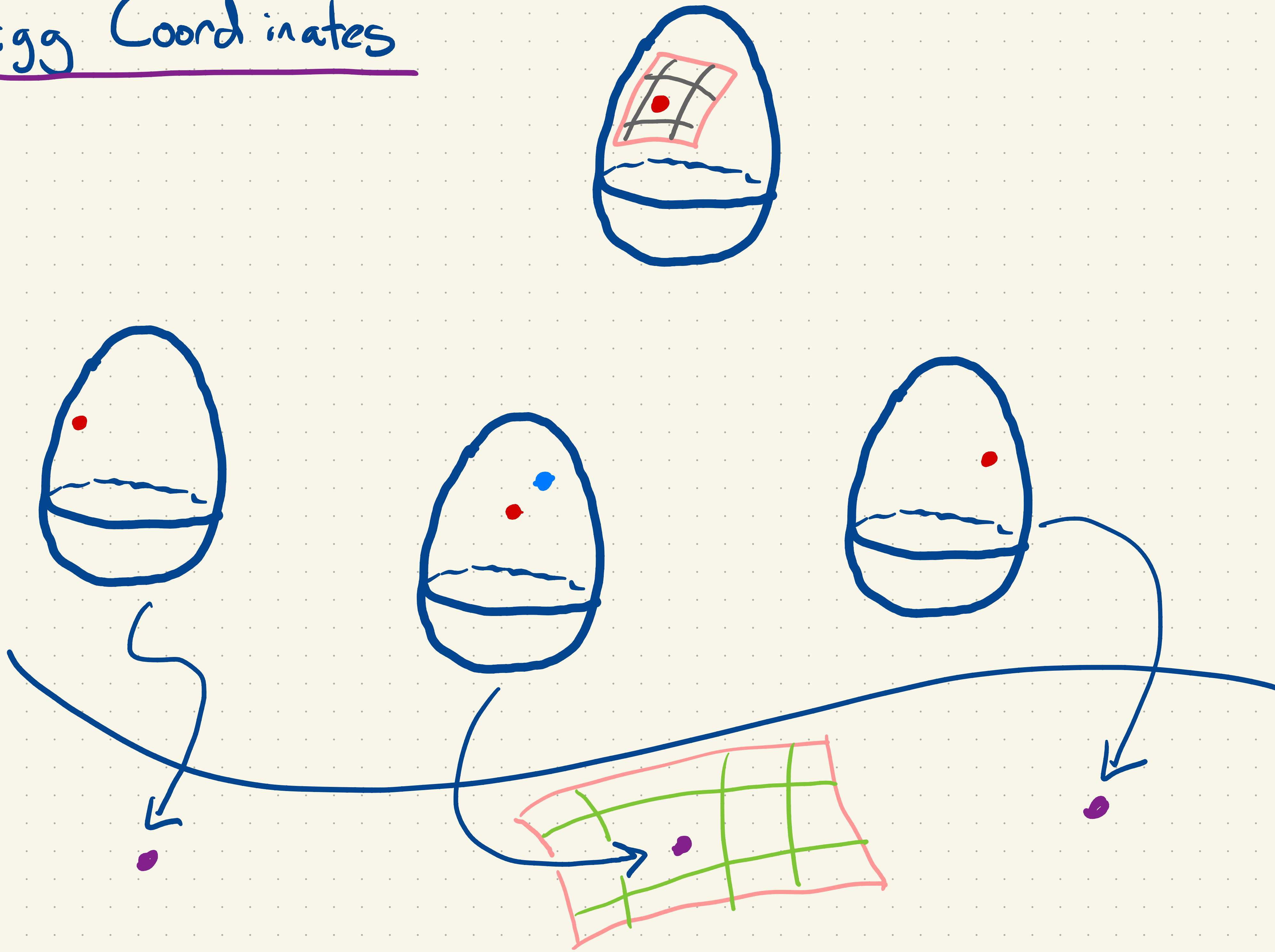
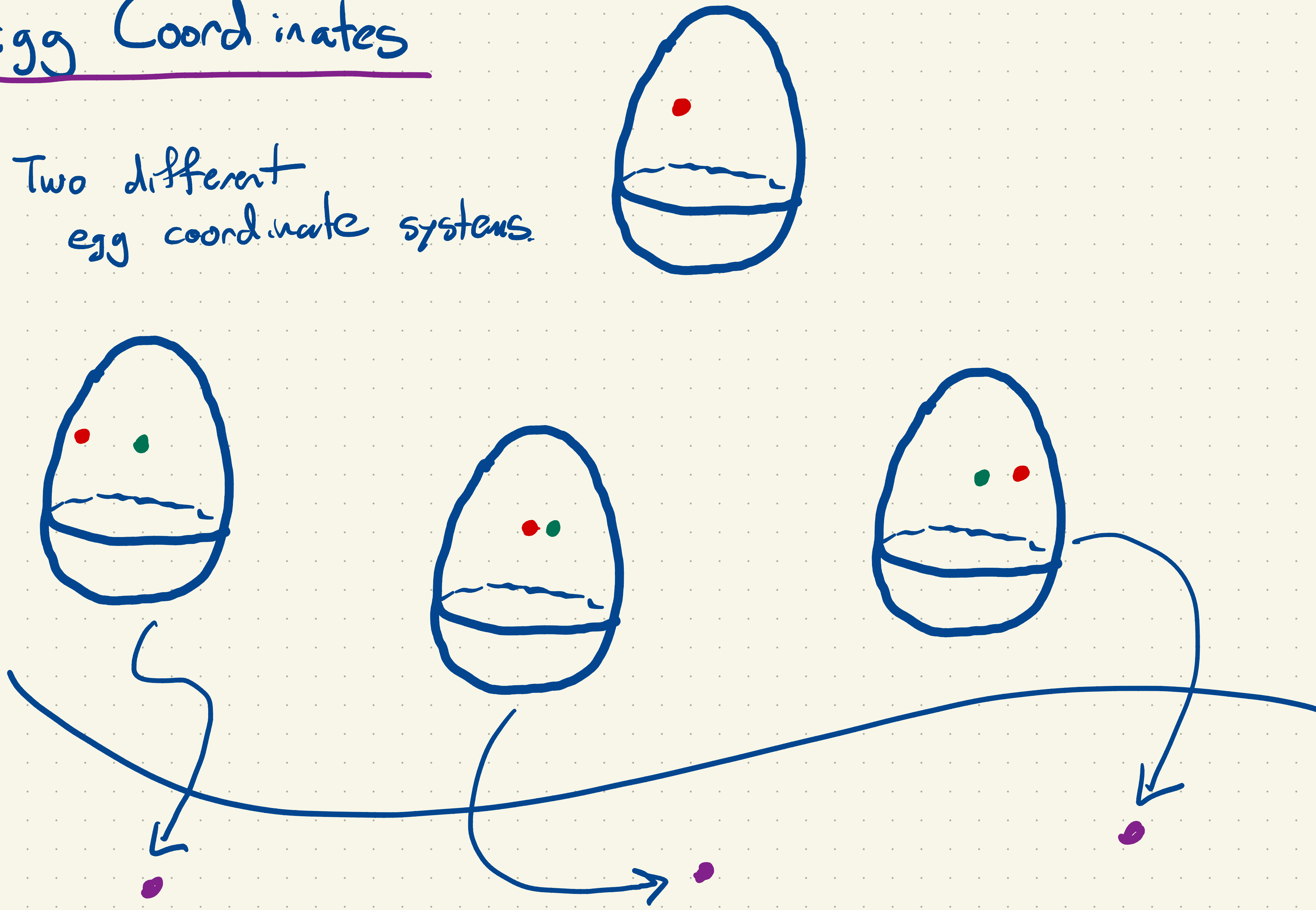


Egg Coordinates

- Two different egg coordinate systems.



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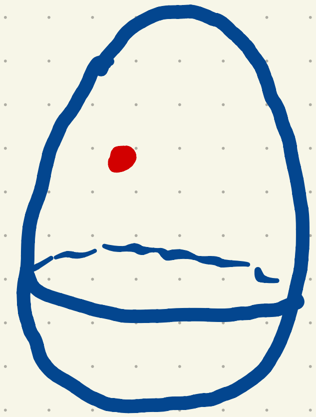
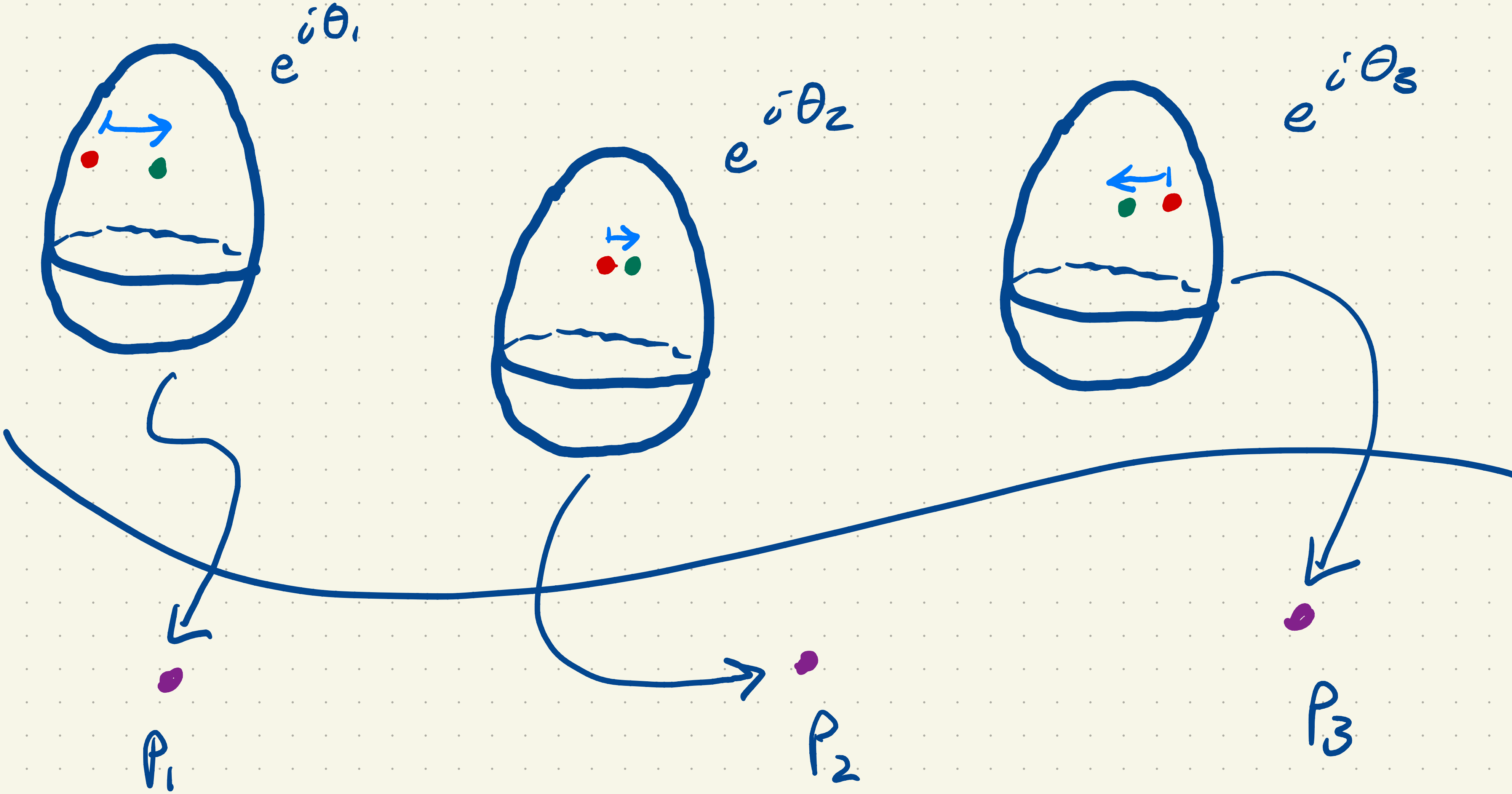
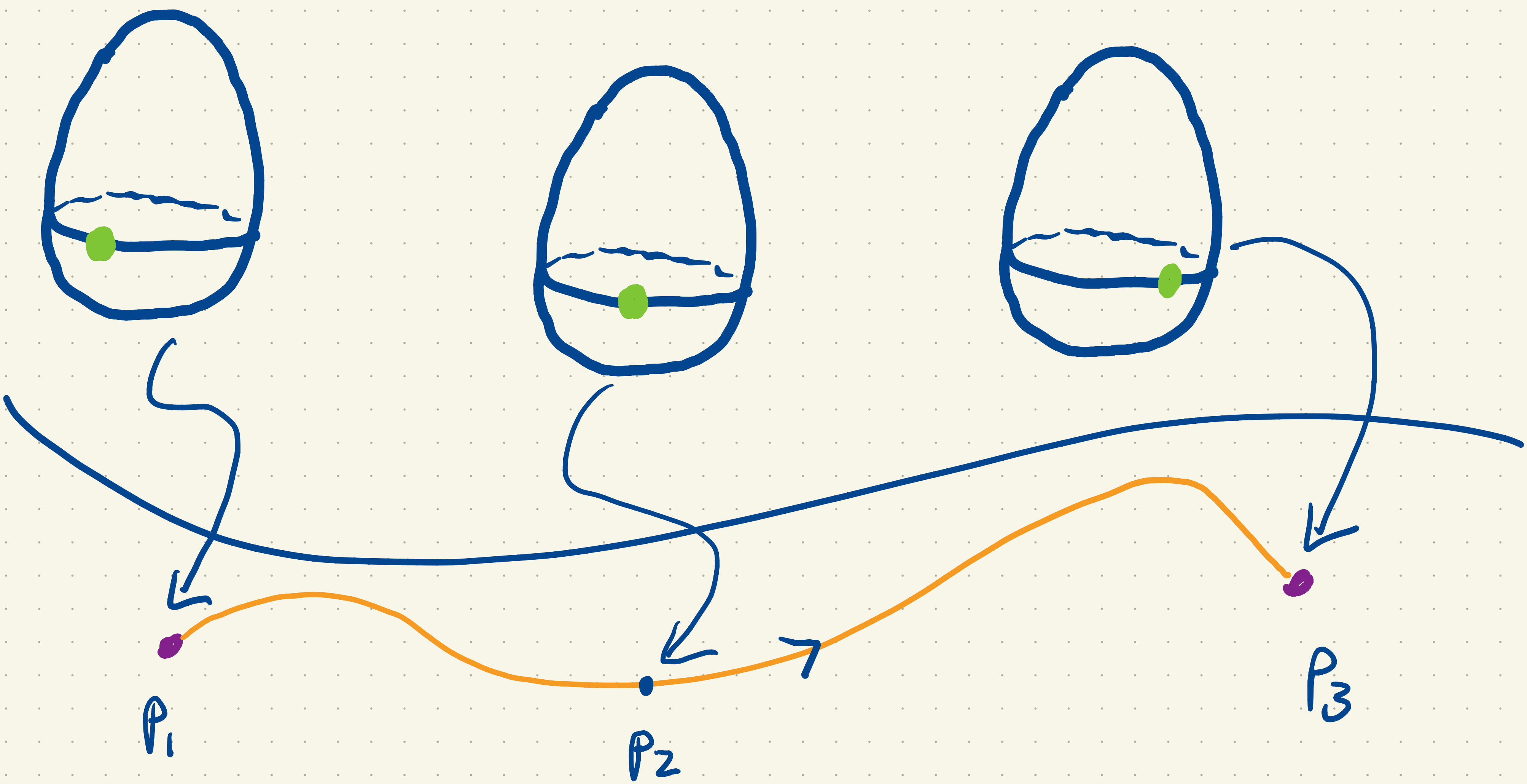




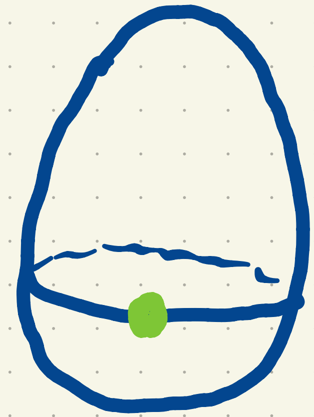
Egg Dragging

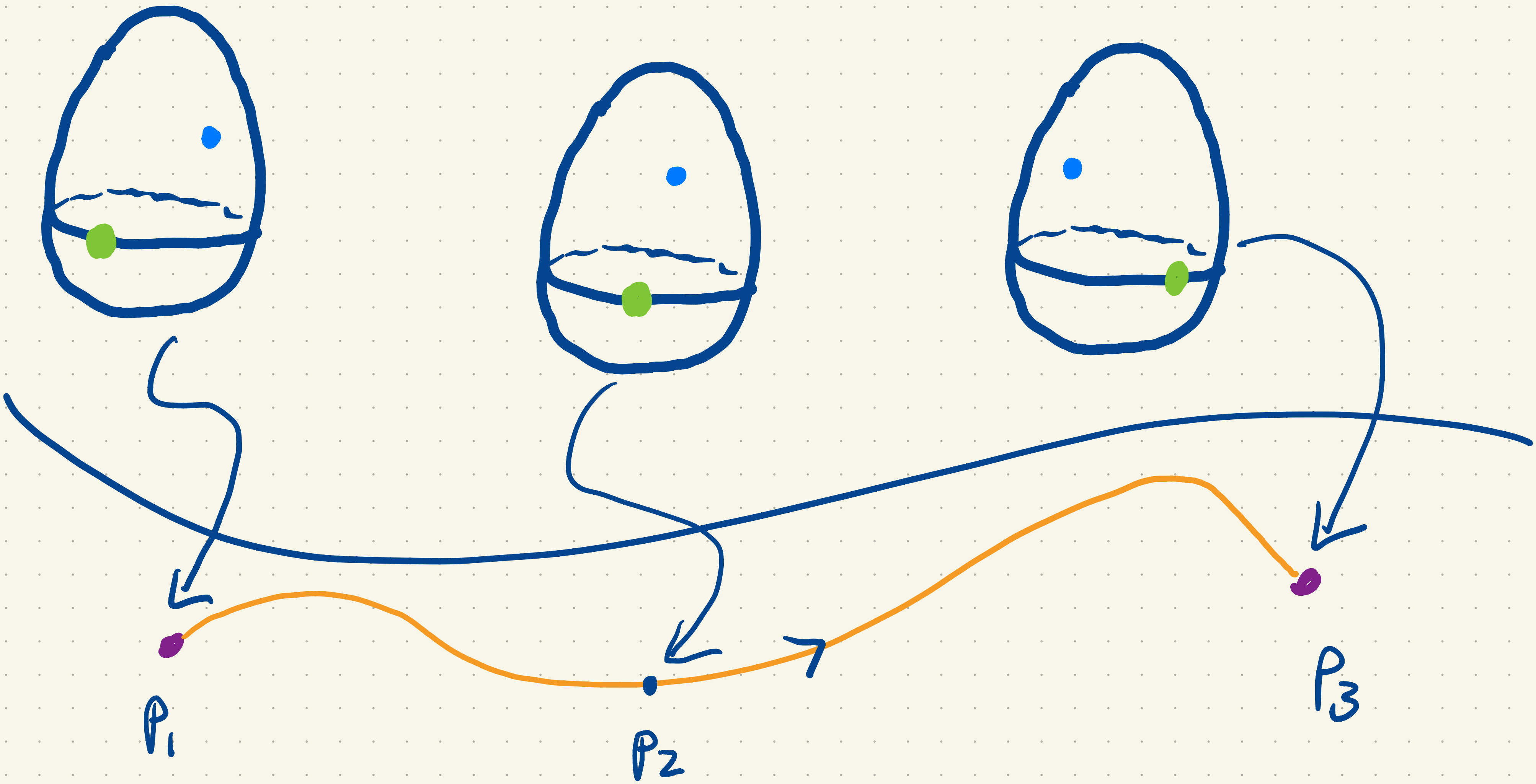
- Is an additional structure beyond the egg structure.



Egg Dragging

- It lets you measure change in the egg along a curve.





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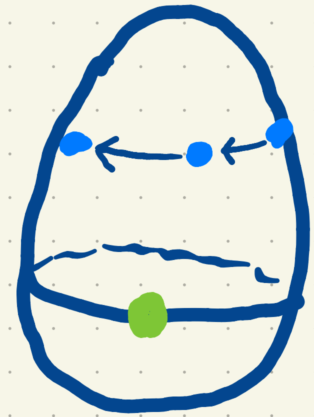
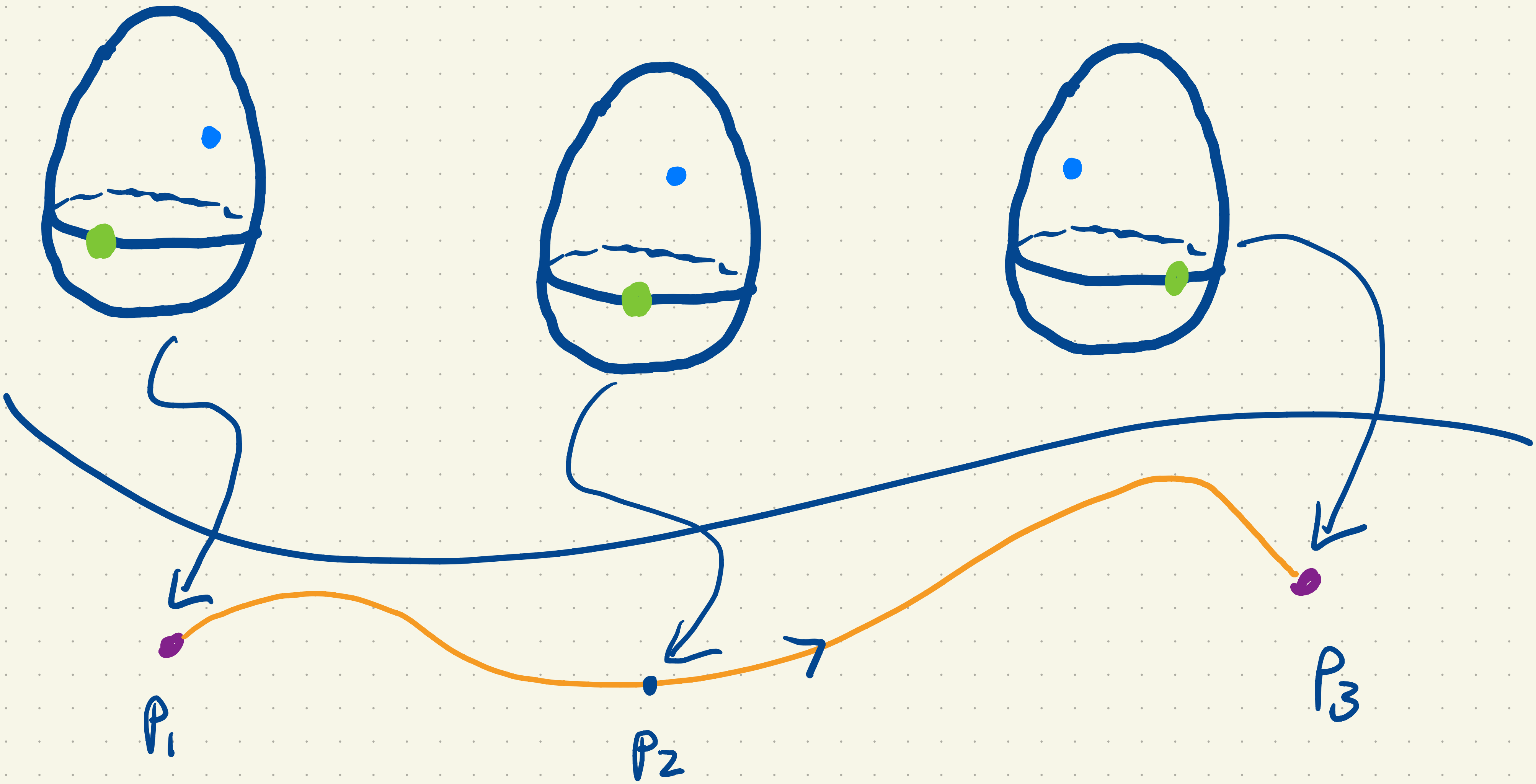
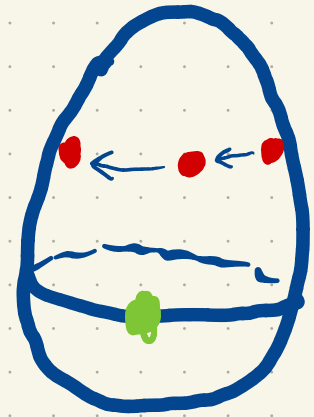
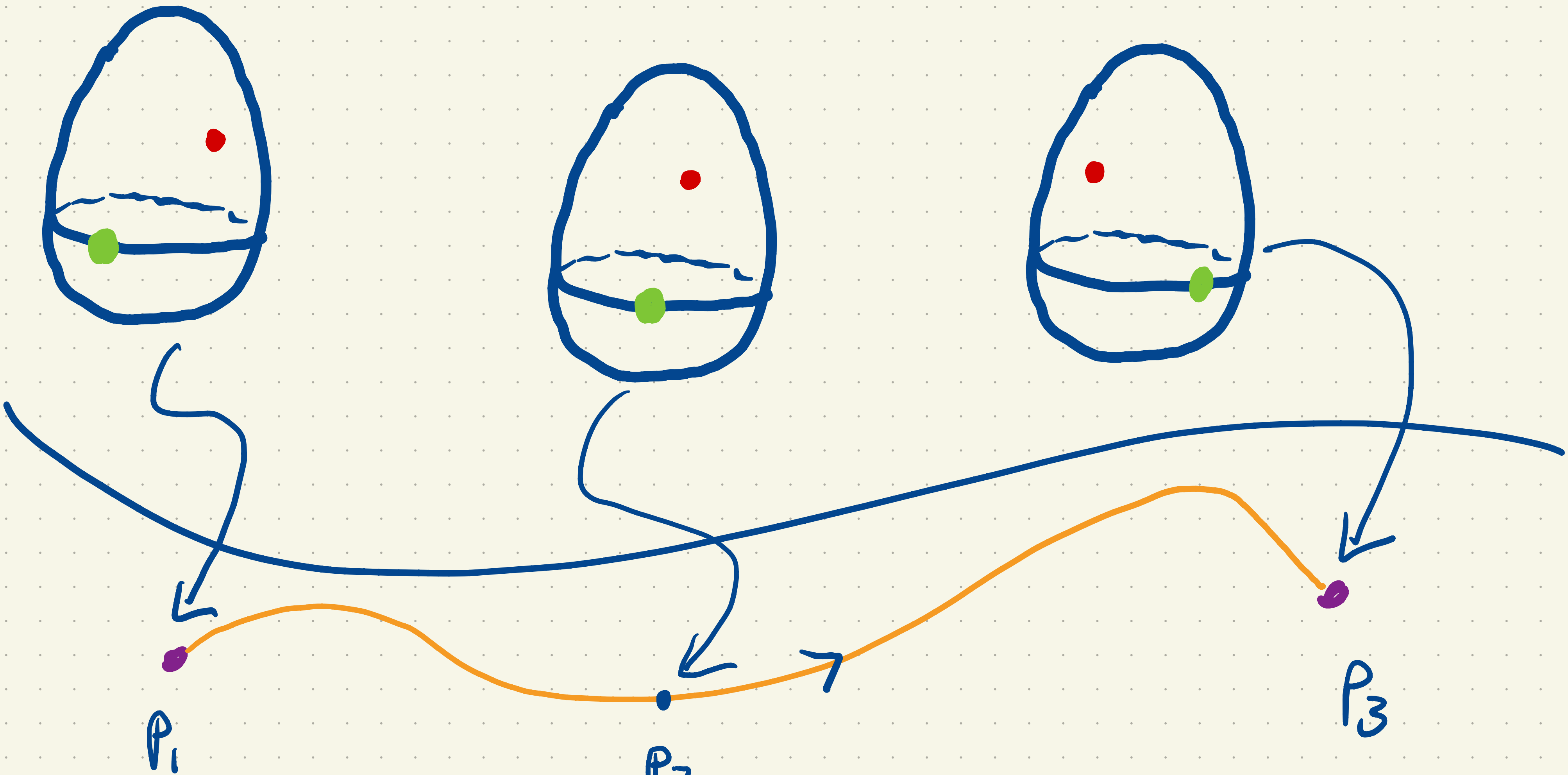




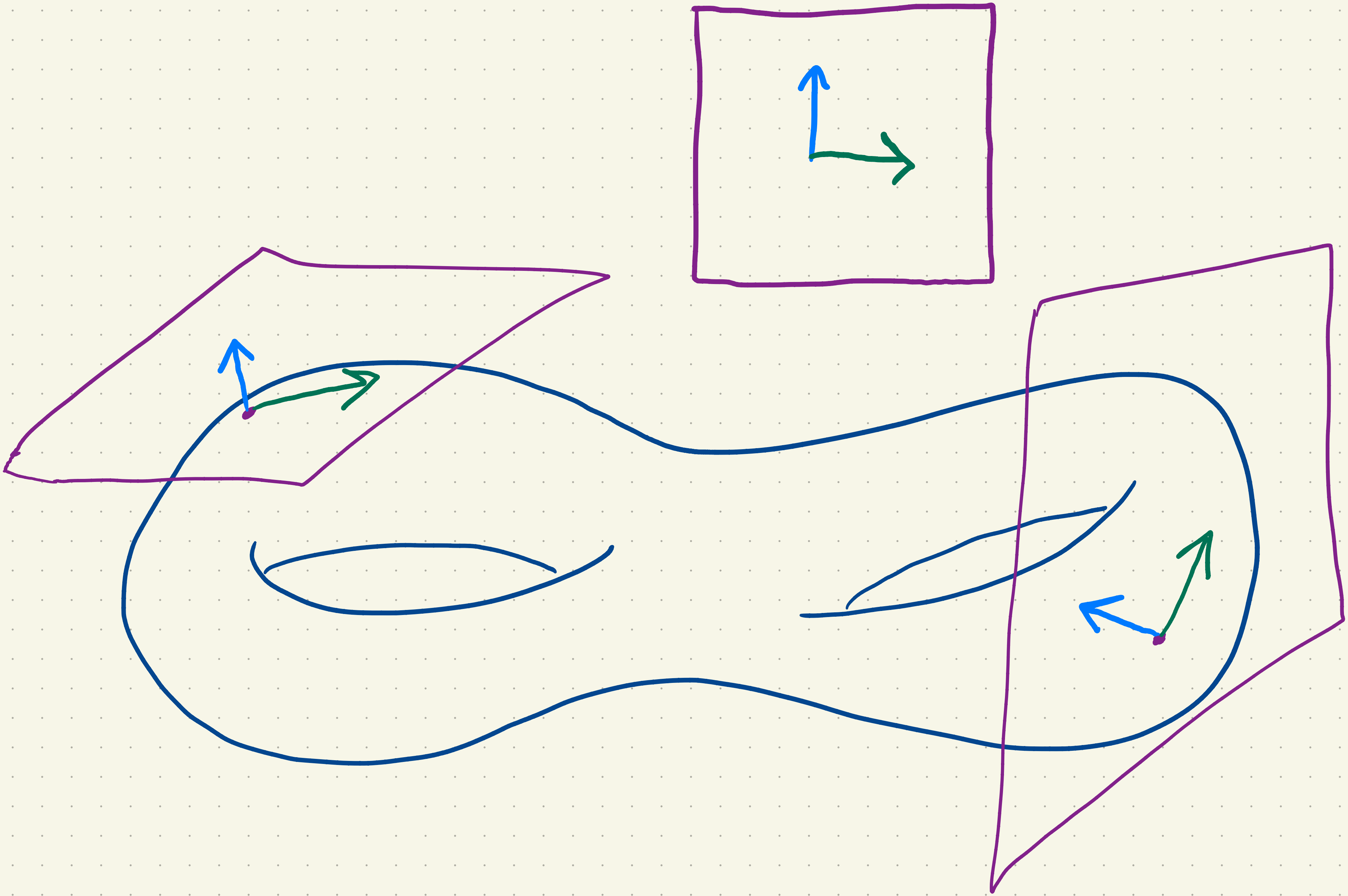
Egg Dragging

- If you pick egg coordinates it tells you "rate of rotation" of coordinates along the curve





An Egg Can be a Plane



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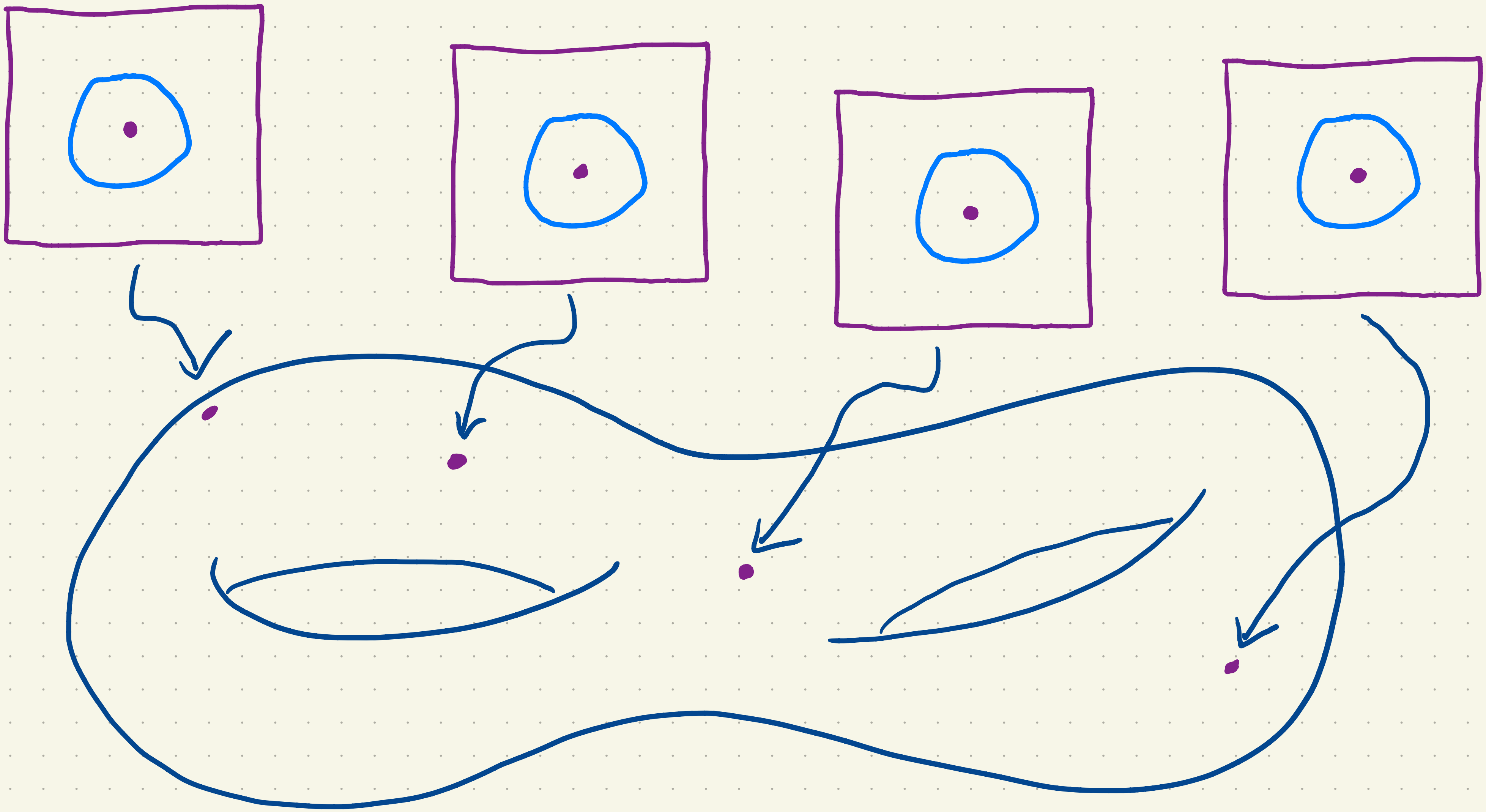
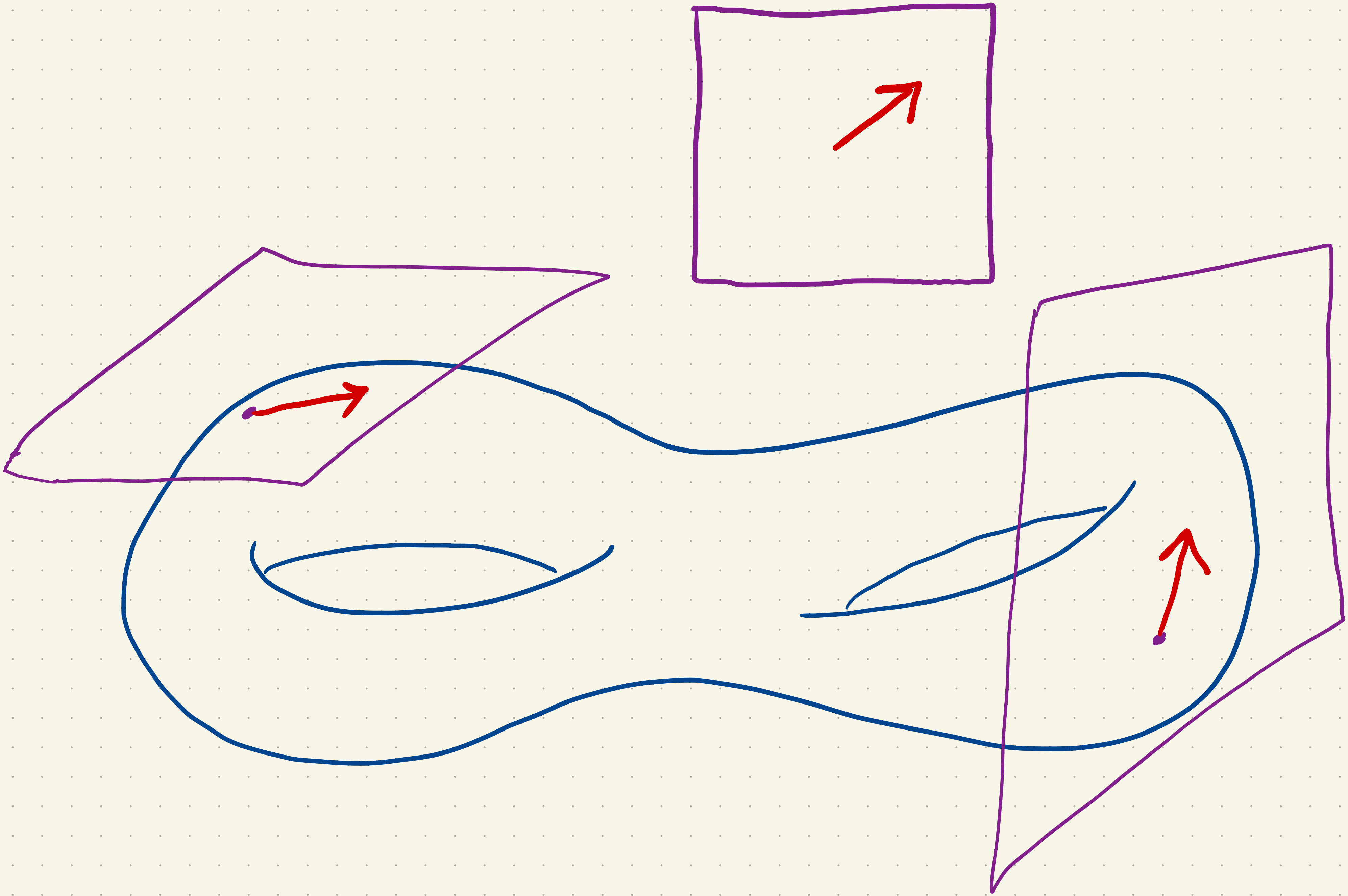


Egg Coordinates



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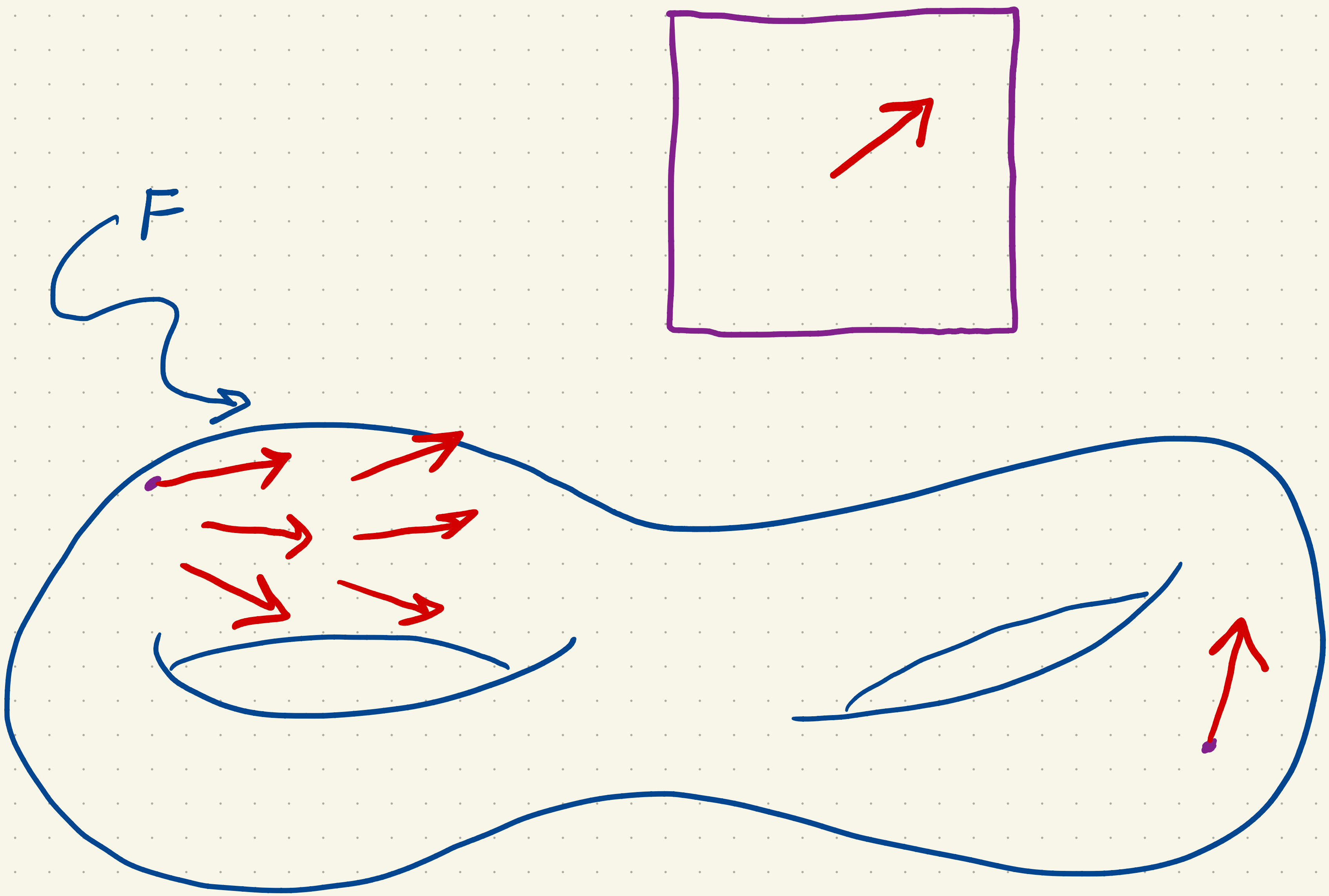

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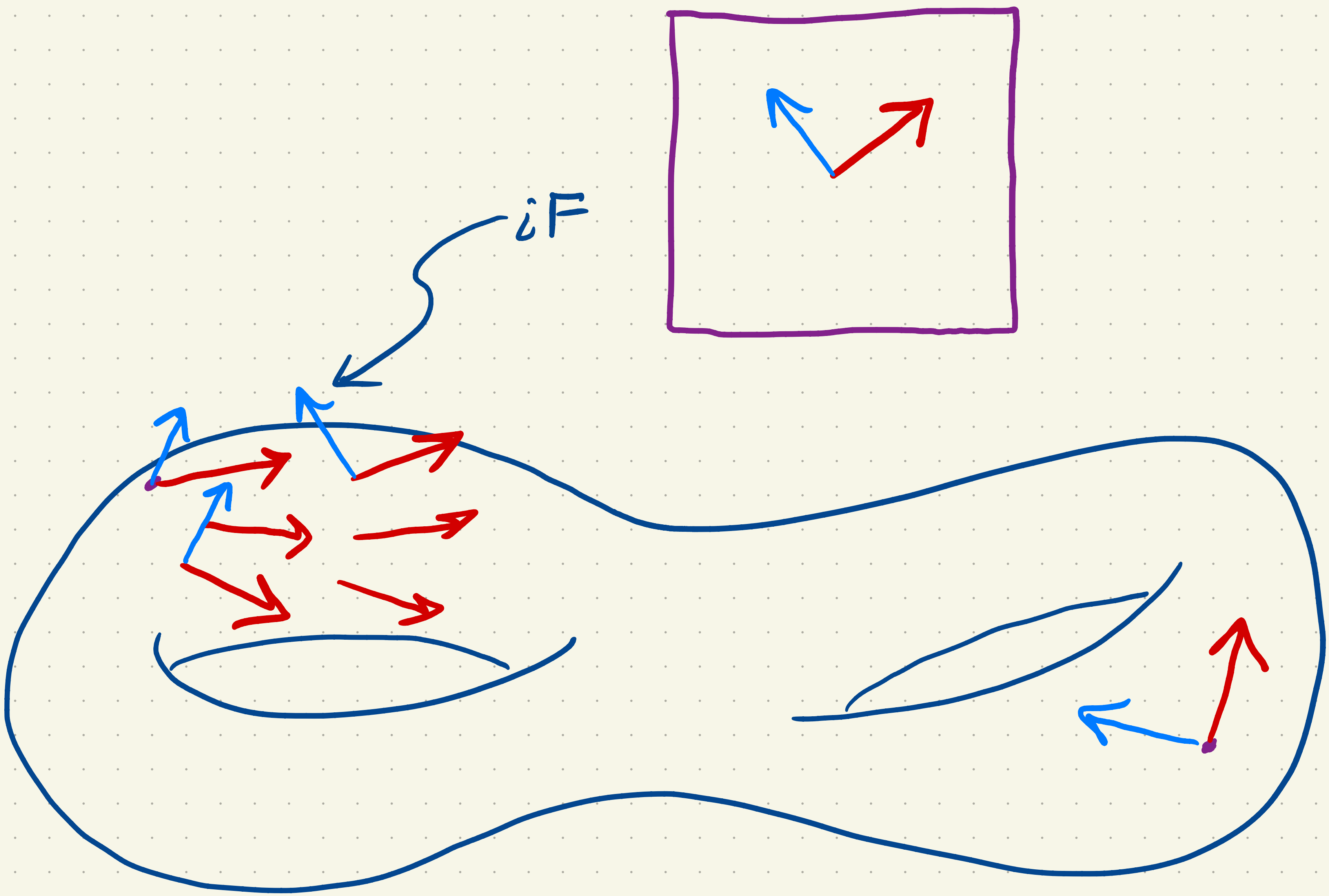

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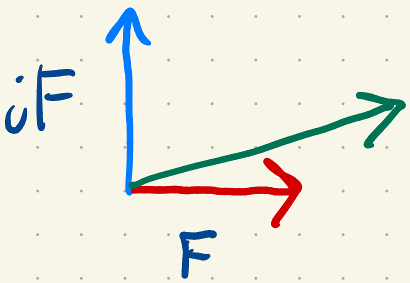
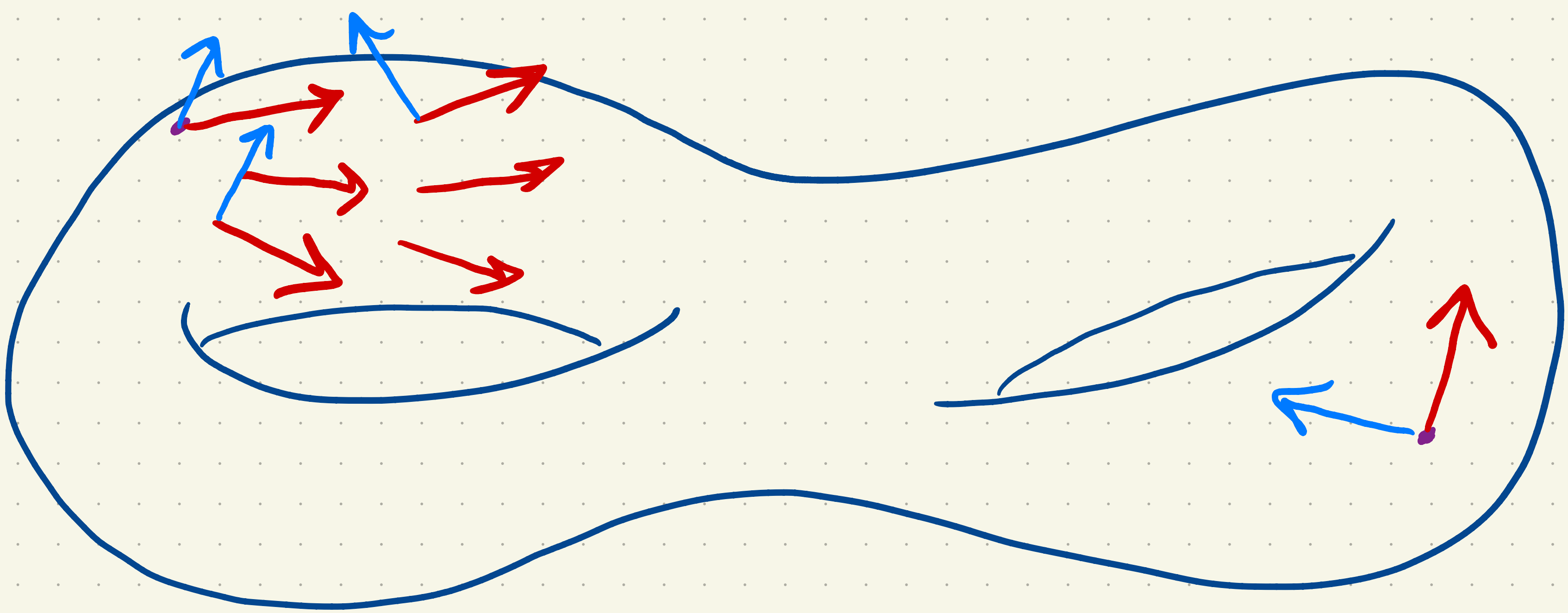


$$Z = aF + ibF$$
$$= zF$$

representation of vector field via $z \in \mathbb{C}$



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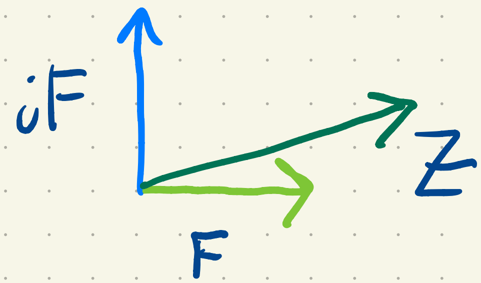
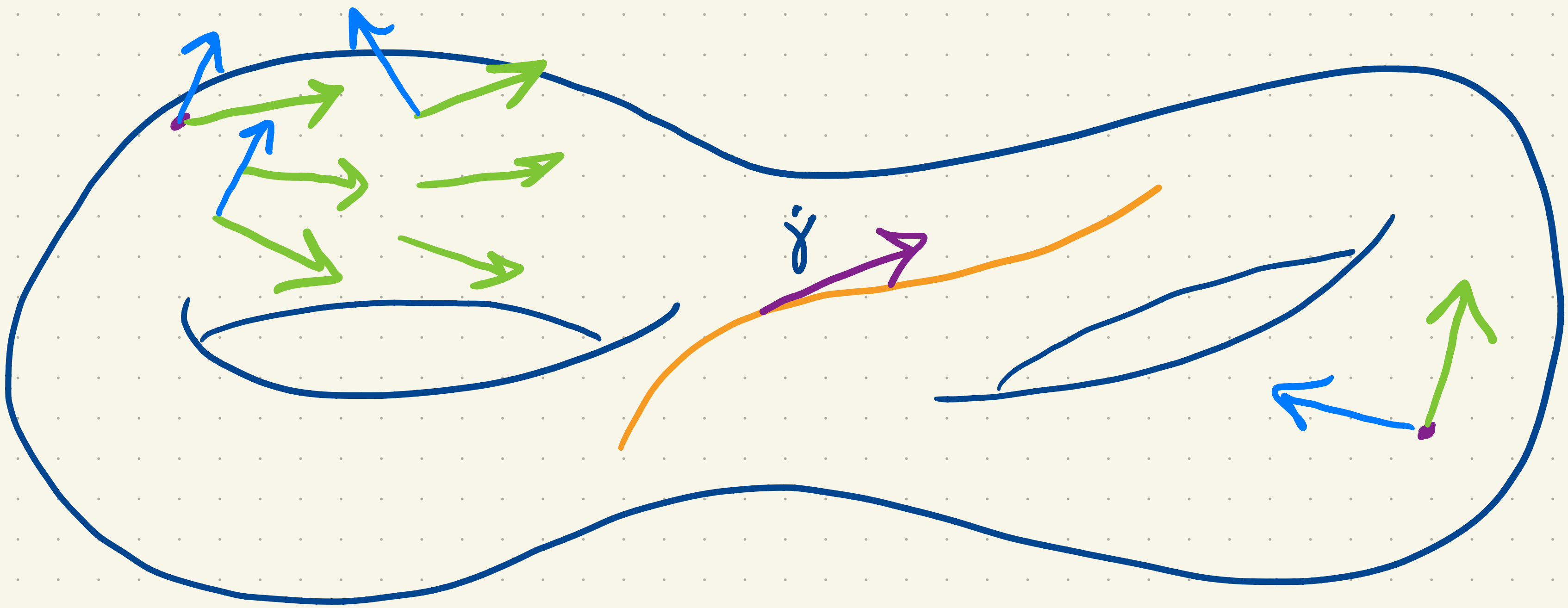
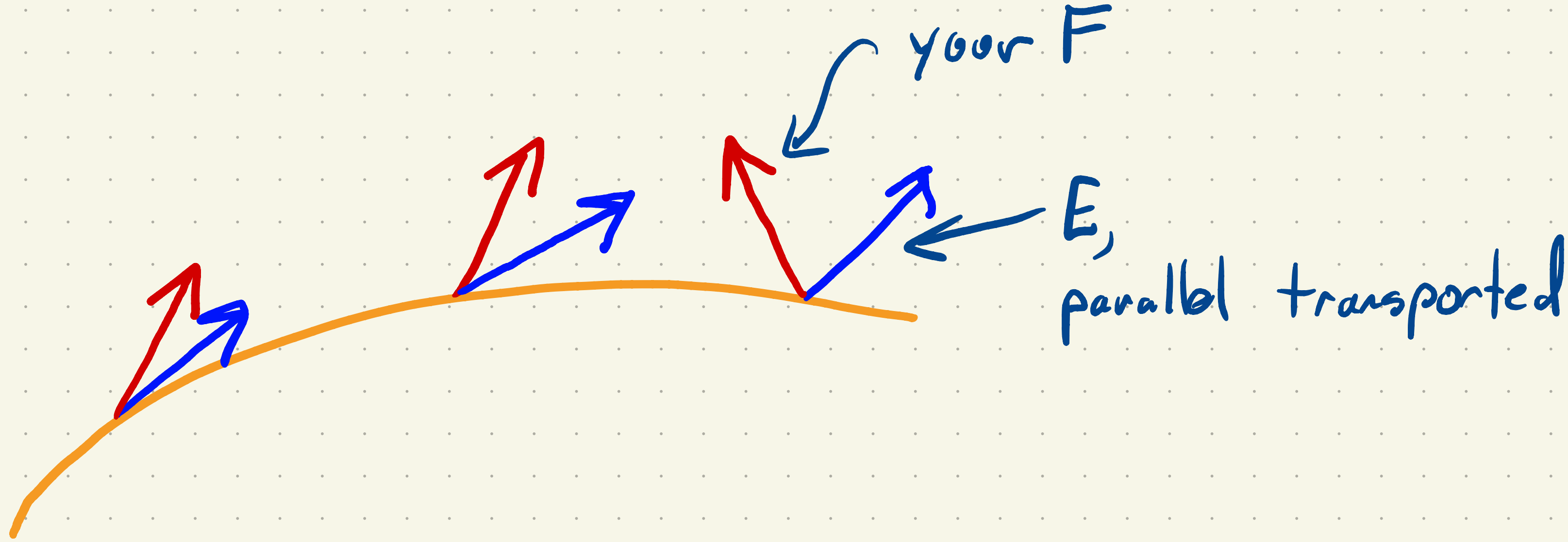


- How to compute

$$\nabla_{\dot{\gamma}} Z \ ?$$

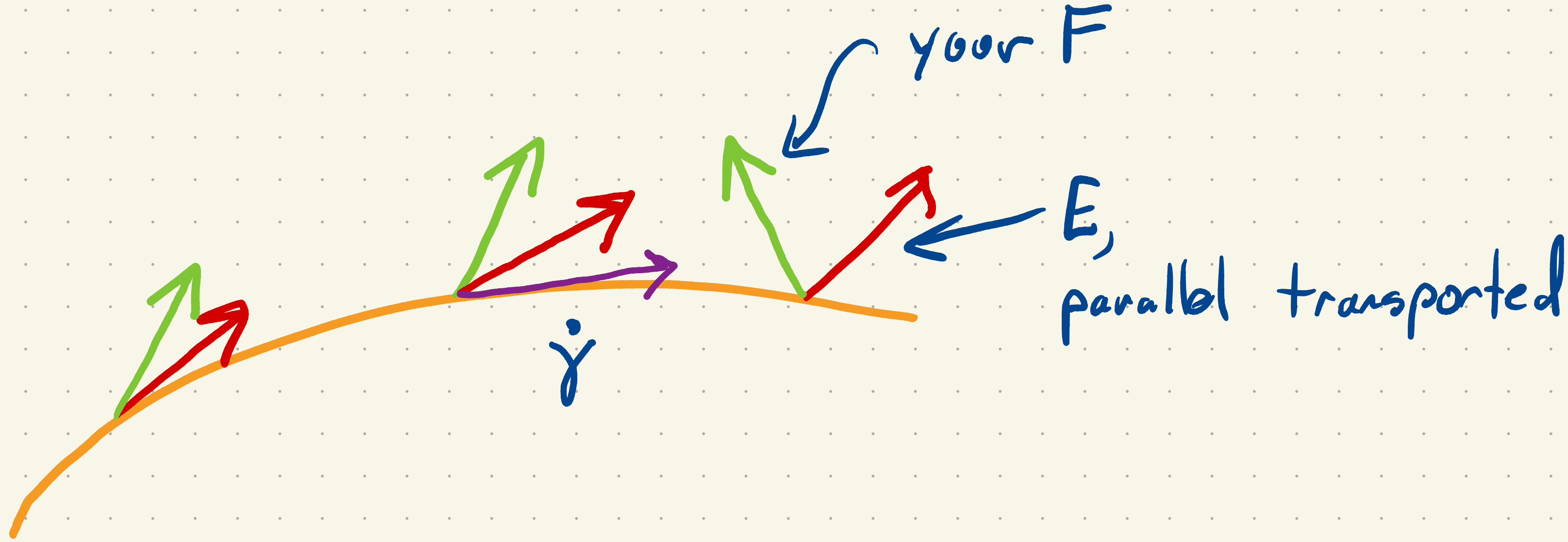


Connection 1-form



$$F = e^{i\theta} E$$

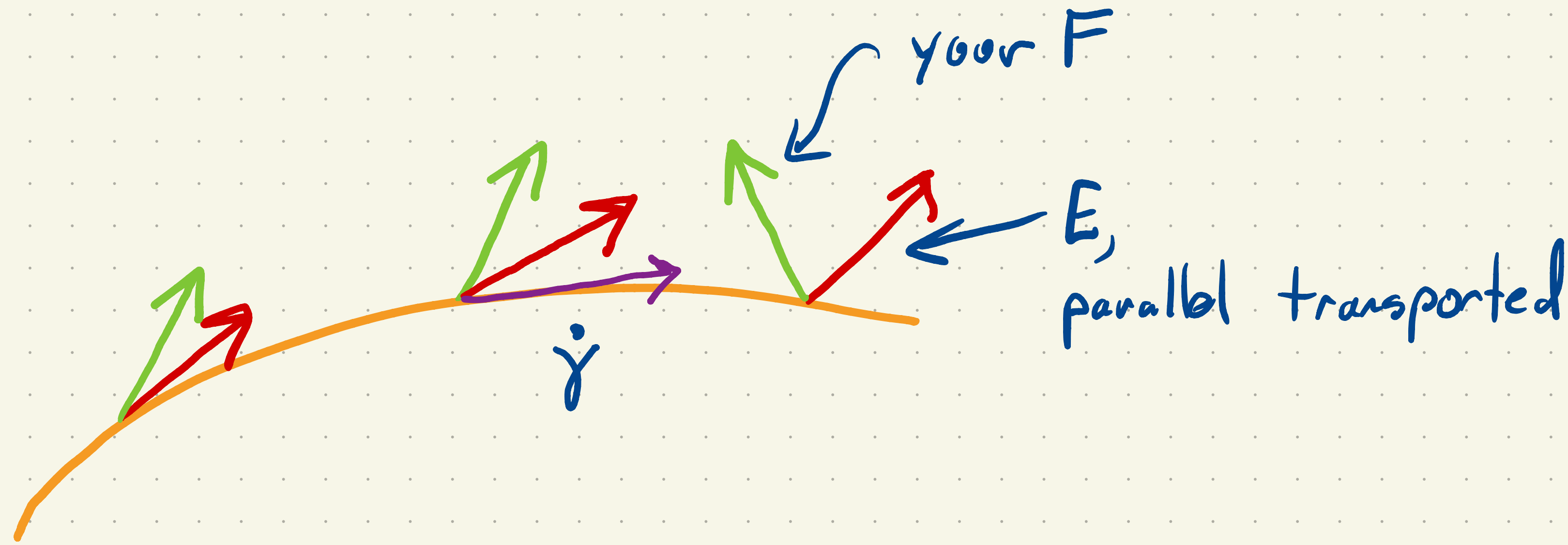
Connection 1-form



$$F = e^{i\theta} E$$

$$\nabla_{\dot{\gamma}} F = i e^{i\theta} \dot{\theta} E = i \dot{\theta} F$$

Connection 1-form



$$F = e^{i\theta} E$$

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$$A[\dot{\gamma}] = \dot{\theta}$$

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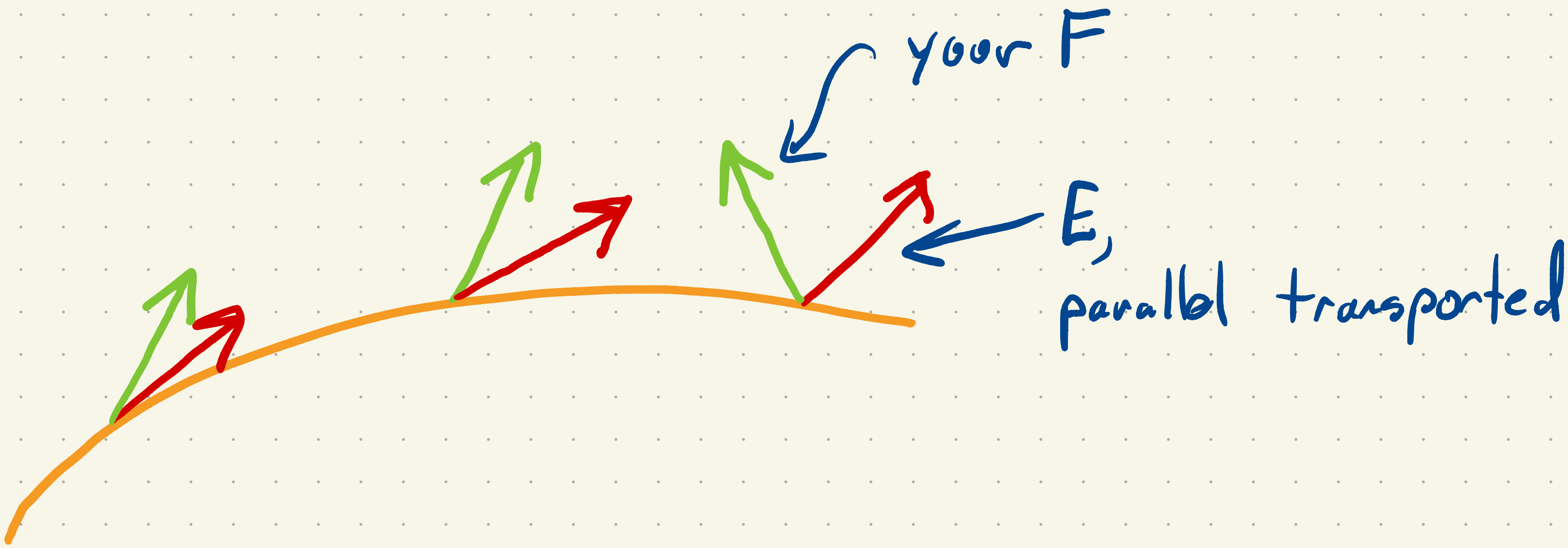


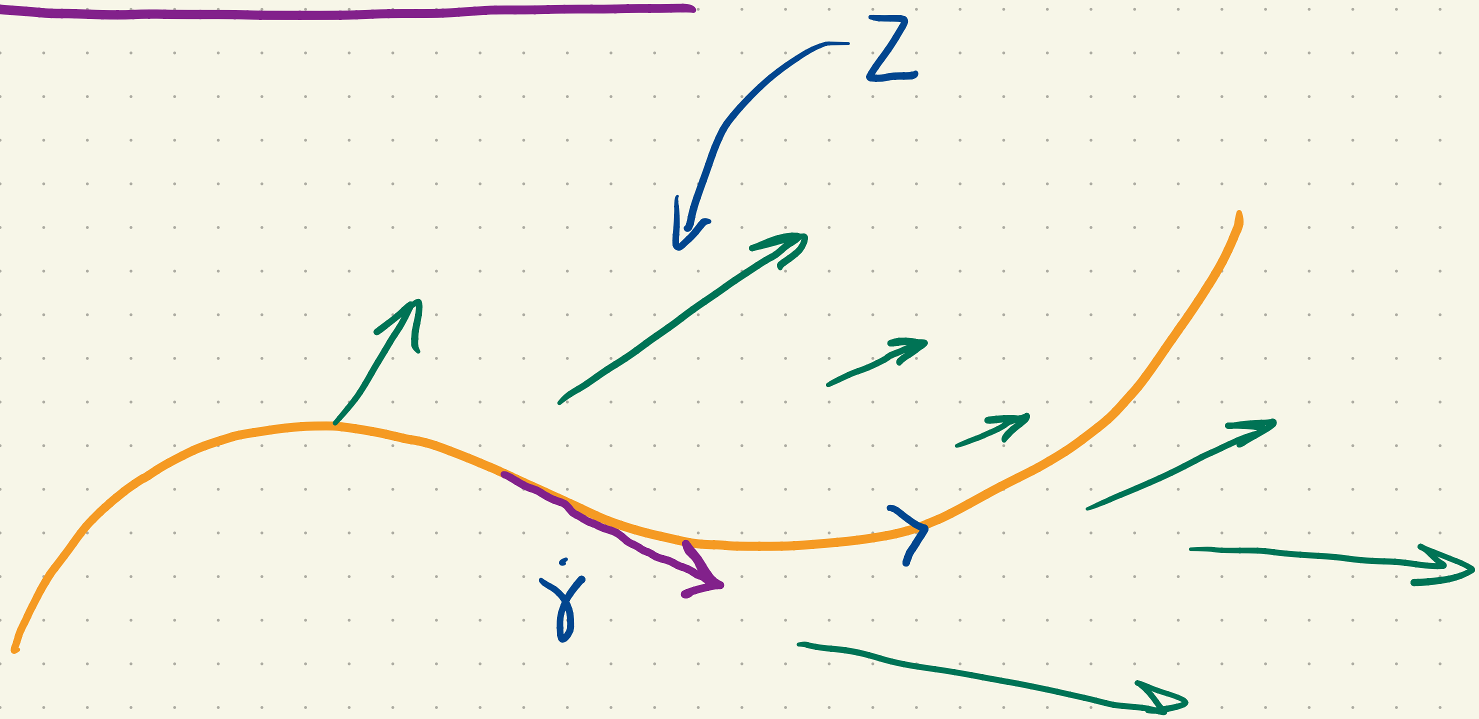
$$F = e^{i\theta} E$$

$$\nabla_{\dot\gamma} F = i e^{i\theta} \dot\theta E = i \dot\theta F$$

$$A[\dot\gamma] = \dot\theta$$

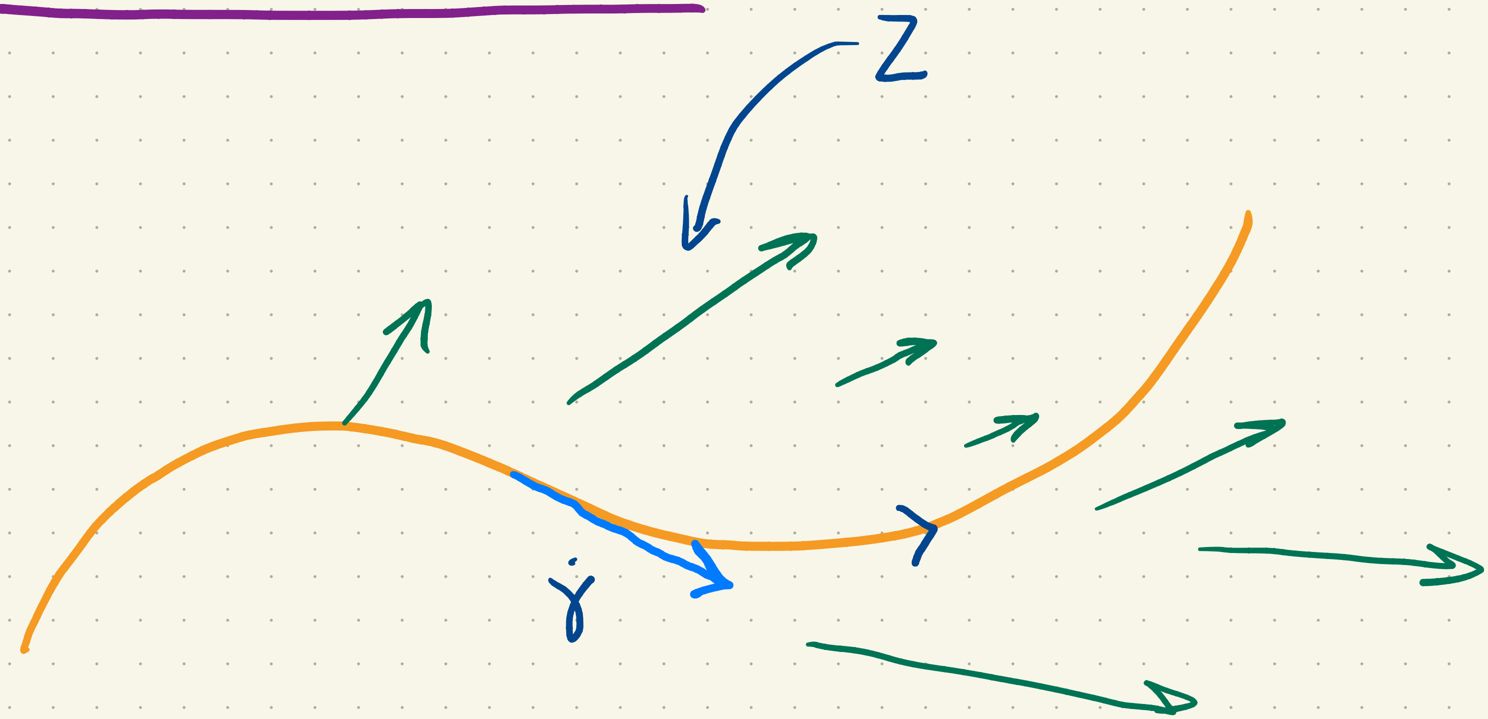
$$\nabla_X F = i A[X] F$$

Derivatives Redux



$$Z = zF$$

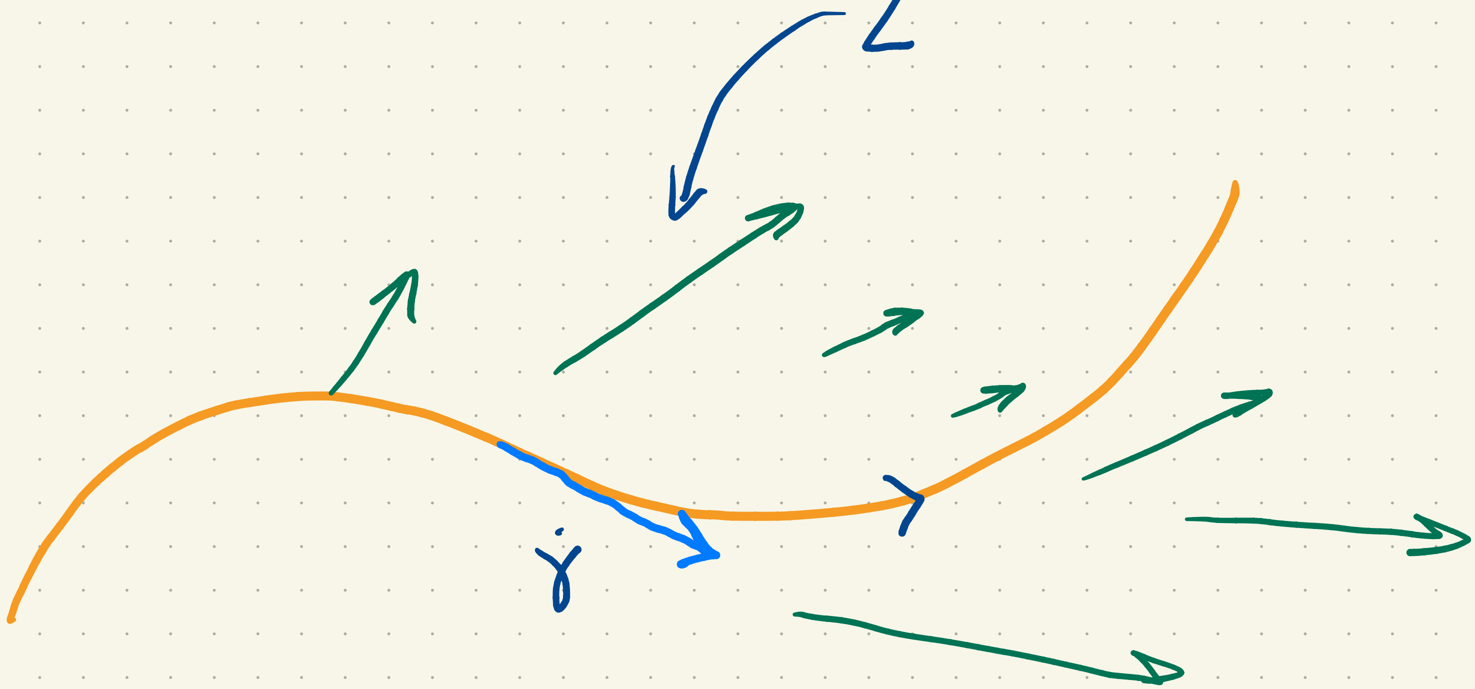
Derivatives Redux



$$Z = z F$$

$$\nabla_{\dot{\gamma}} Z = (\nabla_{\dot{\gamma}} z) F + z \nabla_{\dot{\gamma}} F$$

Derivatives Redux



$$Z = z F$$

$$\nabla_{\dot{\gamma}} Z = \left(\nabla_{\dot{\gamma}} z\right) F + z \nabla_{\dot{\gamma}} F$$

$$= \left(dz[\dot{\gamma}] + z i A[\dot{\gamma}]\right) F$$

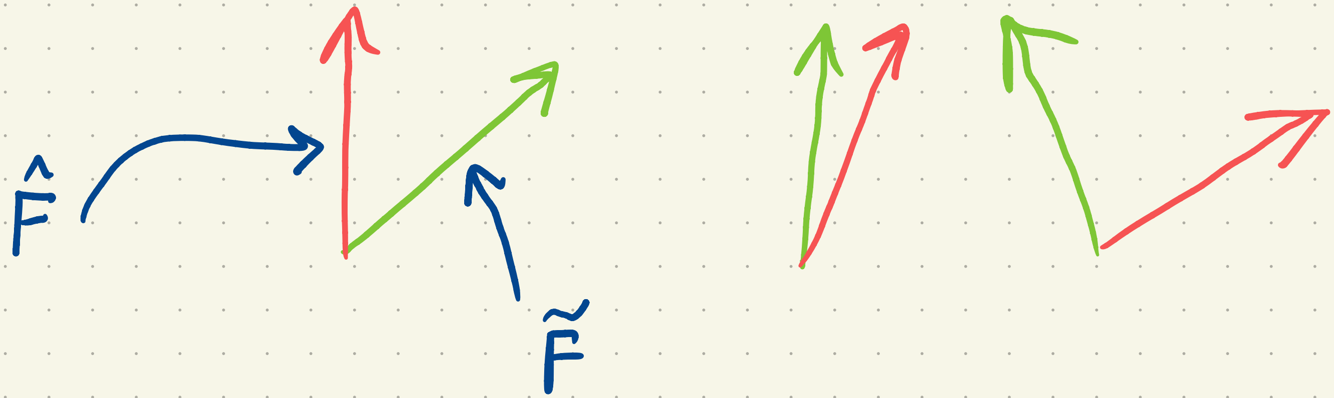
Derivative Summary

- Pick a local frame F
- Compute connection 1-form A
  (relative to F)

$$\nabla_{\dot\gamma} F = i\dot\theta F = iA(\dot\gamma)F$$

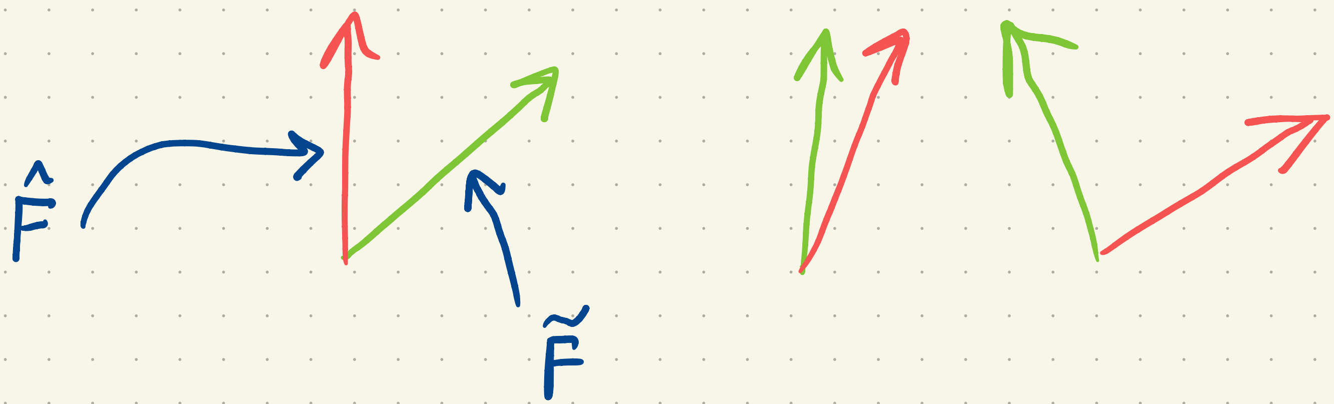
- Represent the vector field Z as zF
- $\nabla_X Z = \left[dz(X) + iA(X)\right]F$

Change of Frame



$$\hat{F} = e^{i\psi}\tilde{F}$$

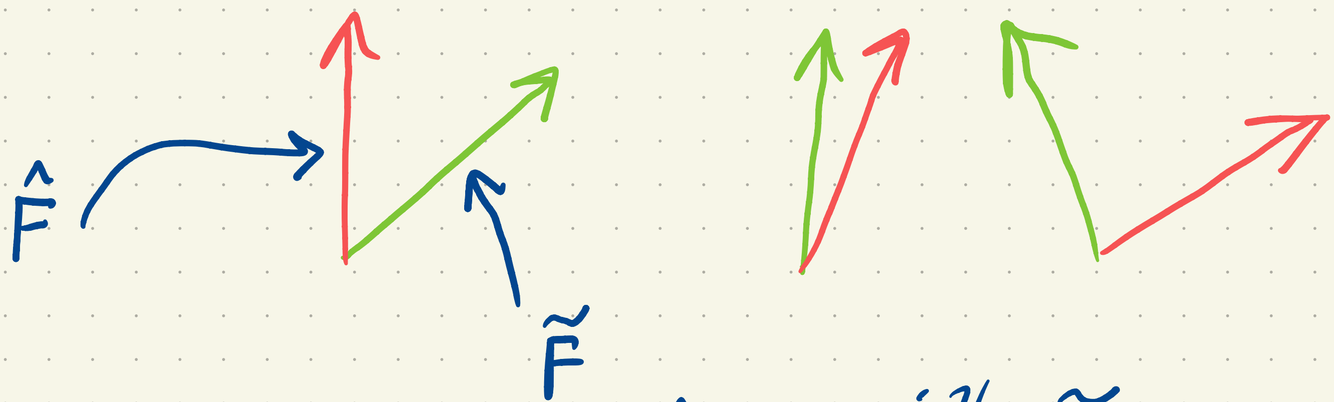
Change of Frame



$$\hat{F} = e^{i\psi}\,\tilde{F}$$

$$\nabla_X \hat{F} = i\,\hat{A}[X]\,\hat{F}$$

Change of Frame

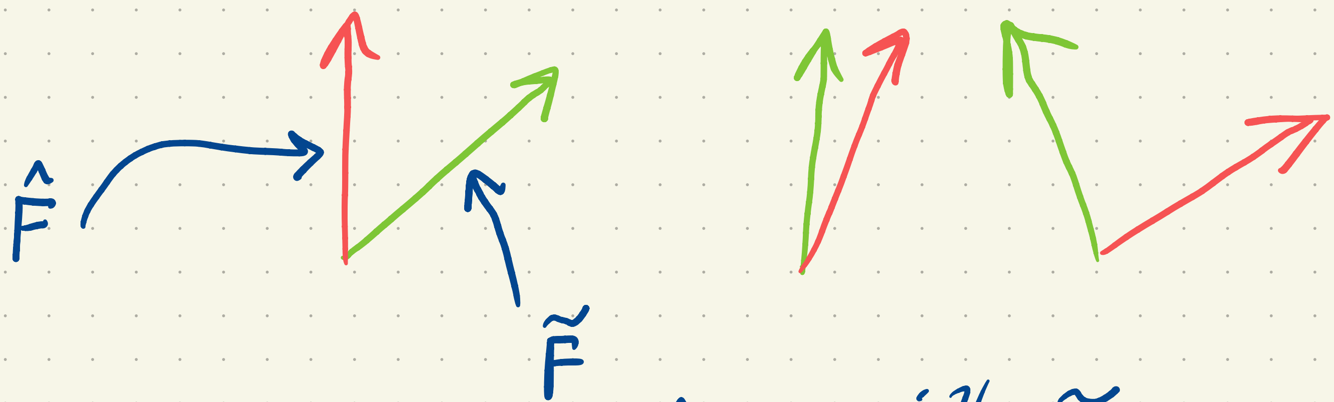


$$\hat{F} = e^{i\psi}\tilde{F}$$

$$\nabla_X \hat{F} = i\hat{A}[X]F$$

$$\nabla_X \hat{F} = \nabla_X\left(e^{i\psi}\tilde{F}\right)$$

Change of Frame



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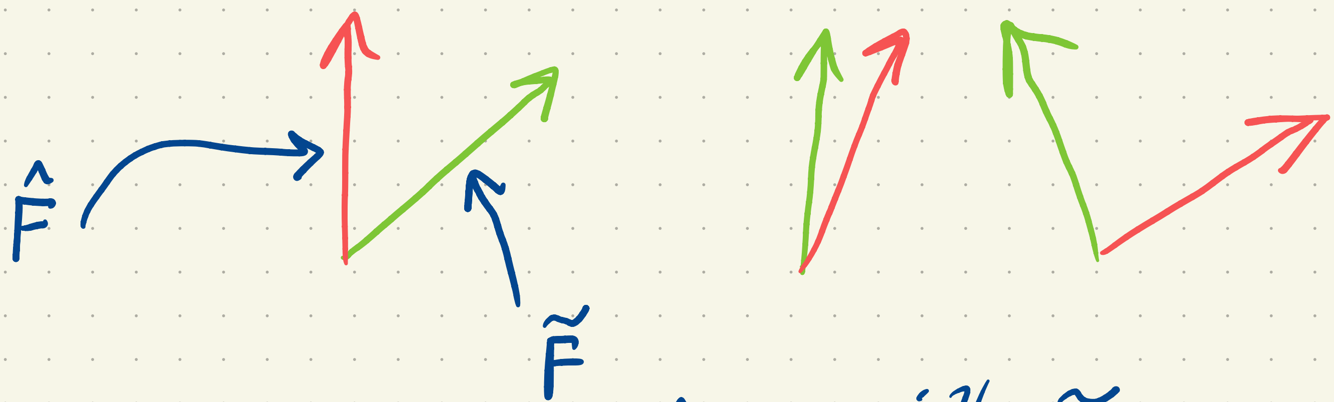
$$= i\,d\psi[X]e^{i\psi}\tilde{F} + e^{i\psi}\,i\tilde{A}[X]\tilde{F}$$

Change of Frame



$$\hat{A} = d\psi + \tilde{A}$$

$$\hat{F} = e^{i\psi}\tilde{F}$$

$$\nabla_X \hat{F} = i\hat{A}[X]F$$

$$\nabla_X \hat{F} = \nabla_X\left(e^{i\psi}\tilde{F}\right)$$

$$= i\,d\psi[X]e^{i\psi}\tilde{F} + e^{i\psi} i\tilde{A}[X]\hat{F}$$

$$= i\left(d\psi[X] + \tilde{A}[X]\right)\hat{F}$$

Change of Frame Summary

	You	Me
Frame "Coordinates"	$\tilde{F}$	$\hat{F} = e^{i\psi}\tilde{F}$
Connection 1-form	$\tilde{A}$	$\hat{A} = d\psi + \tilde{A}$
Vector Field	$\tilde{z}$	$\hat{z} = e^{-i\psi}\tilde{z}$

Very Big Picture

- Spacetime has a local $U(1)$ symmetry.

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- Spacetime has a local $U(1)$ symmetry.
- Maxwell's equations govern a connection.
- After a choice of frame, the connection can be represented by a 1-form A.
- A and $A + d\Psi$ represent the same connection
- The electric and magnetic fields are related to dA.

Homogeneous Maxwell Equations

Inertial coordinates of SR: (t, x^1, x^2, x^3), $c = 1$

$$A = \phi\, dt - A^i dx^i \quad (\text{implicit sum})$$

$$dA = -E^i dx^i \wedge dt - B_{ij}\, dx^i \wedge dx^j$$

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$$\vec{B} = (B^1, B^2, B^3) \qquad B^1 = B_{23}, \; B^2 = B_{31}, \; B^3 = B_{12}$$

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$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Homogeneous Maxwell Equations

$$dA = -E^i \, dx^i \wedge dt - B_{ij} \, dx^i \wedge dx^j$$

$$d(A + d\chi) = dA \qquad d^2 = 0$$

Homogeneous Maxwell Equations

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$$d^2 A = 0$$

$$\left(\partial_j E^i - \dot{B}_{ij}\right) dx^i \wedge dx^j \wedge dt$$

$$- \partial_k B_{ij} \, dx^k \wedge dx^i \wedge dx^j \quad = 0$$

Homogeneous Maxwell Equations

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$$B_{12} \rightarrow B^3$$

$$\dot{\vec{B}} = -\vec{\nabla} \times \vec{E}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

Homogeneous Maxwell Equations

$A + d\psi$

$$dA = -E_i\, dx^i \wedge dt - B_{ij}\, dx^i \wedge dx^j$$

$$d^2 A = 0 \qquad d\mathcal{F} = 0$$

$$\downarrow$$

$$\mathcal{F} = dA$$

$$\left(\partial_j E_i - \dot{B}_{ij}\right) dx^i \wedge dx^j \wedge dt - \partial_k B_{ij}\, dx^k \wedge dx^i \wedge dx^j = 0$$

$$\dot{\vec{B}} = -\vec{\nabla} \times \vec{E}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$d\mathcal{F} = 0$

$\Rightarrow ? \; \exists$

A