An Unhelpful Introduction to Electricity & Magnetism

Part II: Symmetry Dragging

Nov 17, 2020

David Maxwell

UAF Mathematics

Previously, on Unhelpful E+M

- Goal: describe E+M as a cousin of GR
  (as a kind of geometric theory)

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- Exterior derivative on functions:

$$T \longmapsto \underbrace{dT}_{\text{covector}} \qquad \int_\gamma dT = T(\gamma(b)) - T(\gamma(a))$$

Plan

- Finish discussion of exterior derivative

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- Geometry of surfaces and symmetry dragging.

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- Slides + video: damaxwell.github.io

Exterior Derivative of a Function

$T(p)$

Exterior Derivative of a Function

$$T(p)$$

p_0

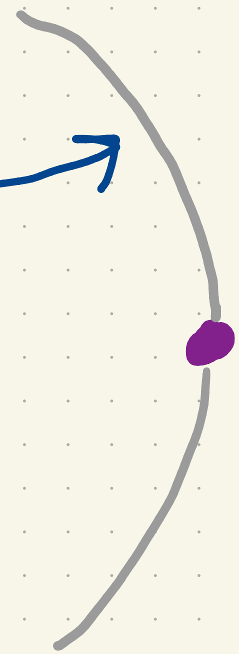
$$\int_{p_0} T = T(p_0)$$

Exterior Derivative of a Function

$T(p)$

$K(y)$

$y \in [c,d]$



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Exterior Derivative of a Function

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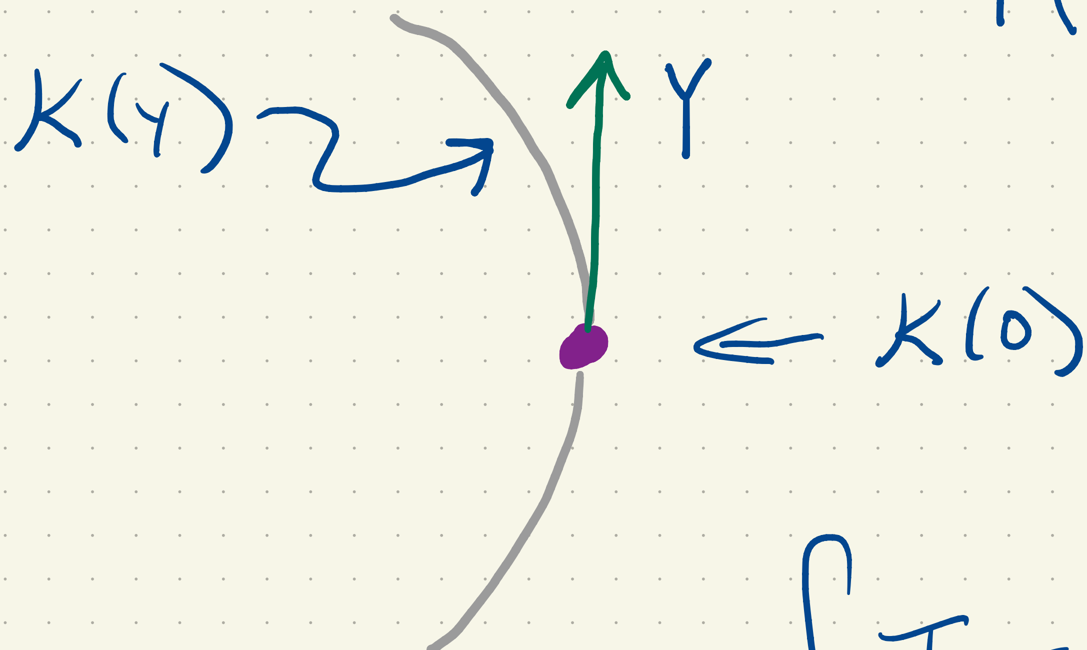
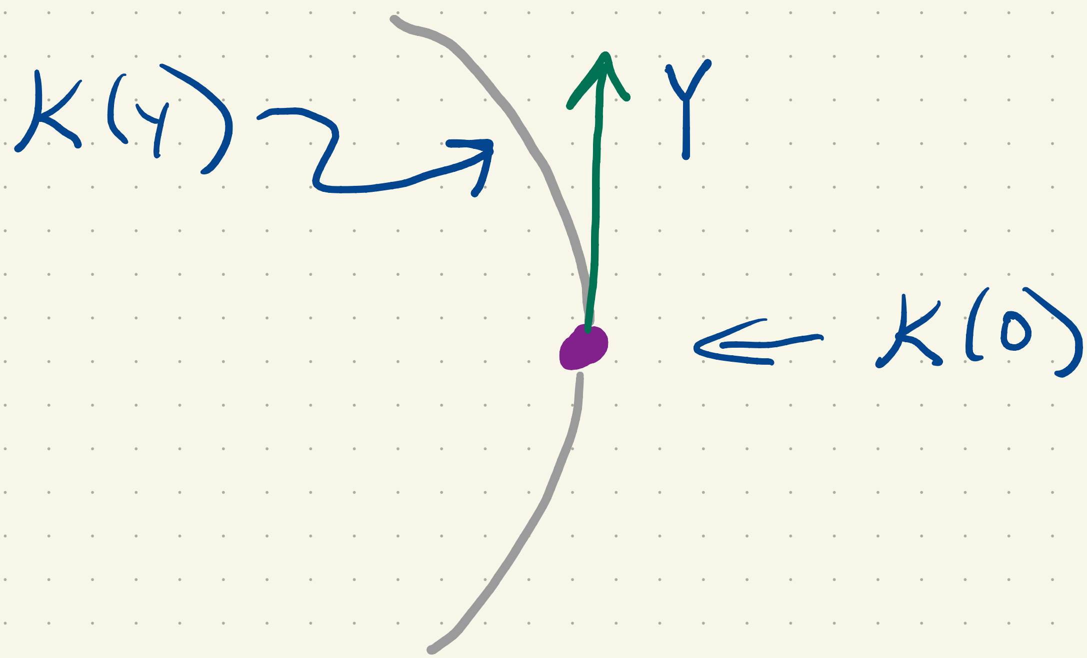


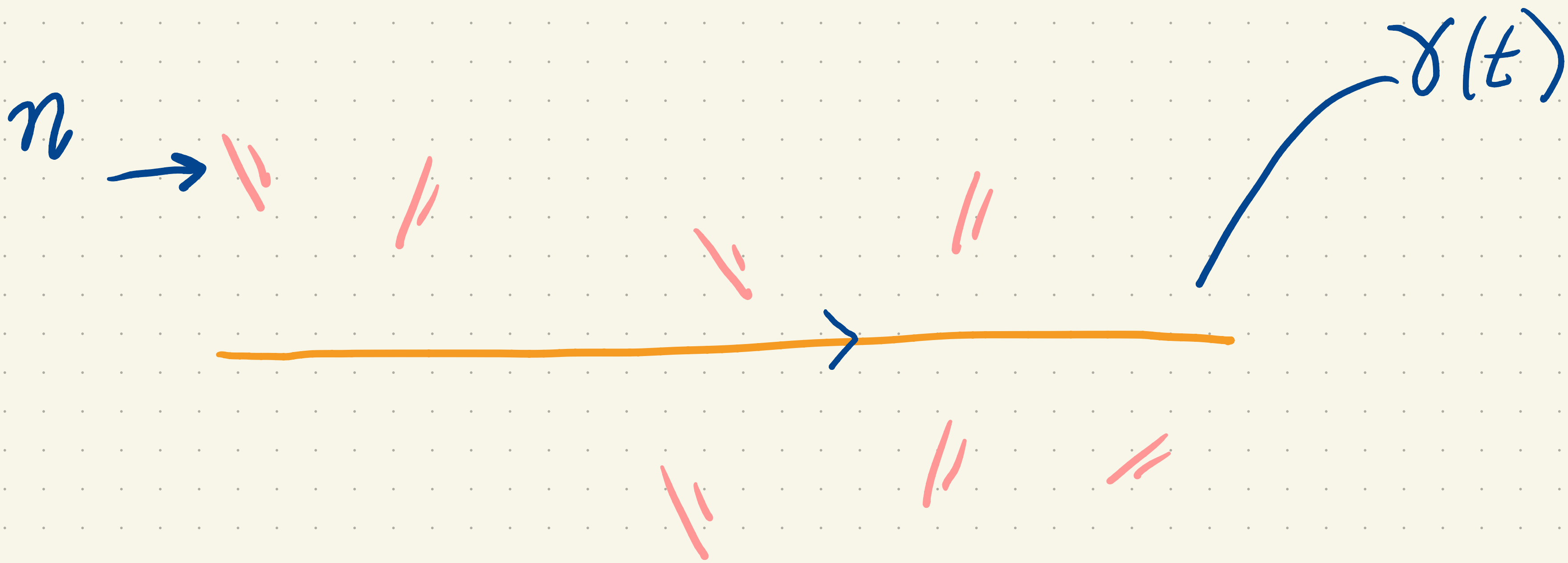
$$\int_p T = T(p)$$

Exterior Derivative of a Function



$$\frac{d}{dy}\bigg|_{y=0} \int T(K(y)) = \int dT[Y]$$

Exterior Derivative of Covectors



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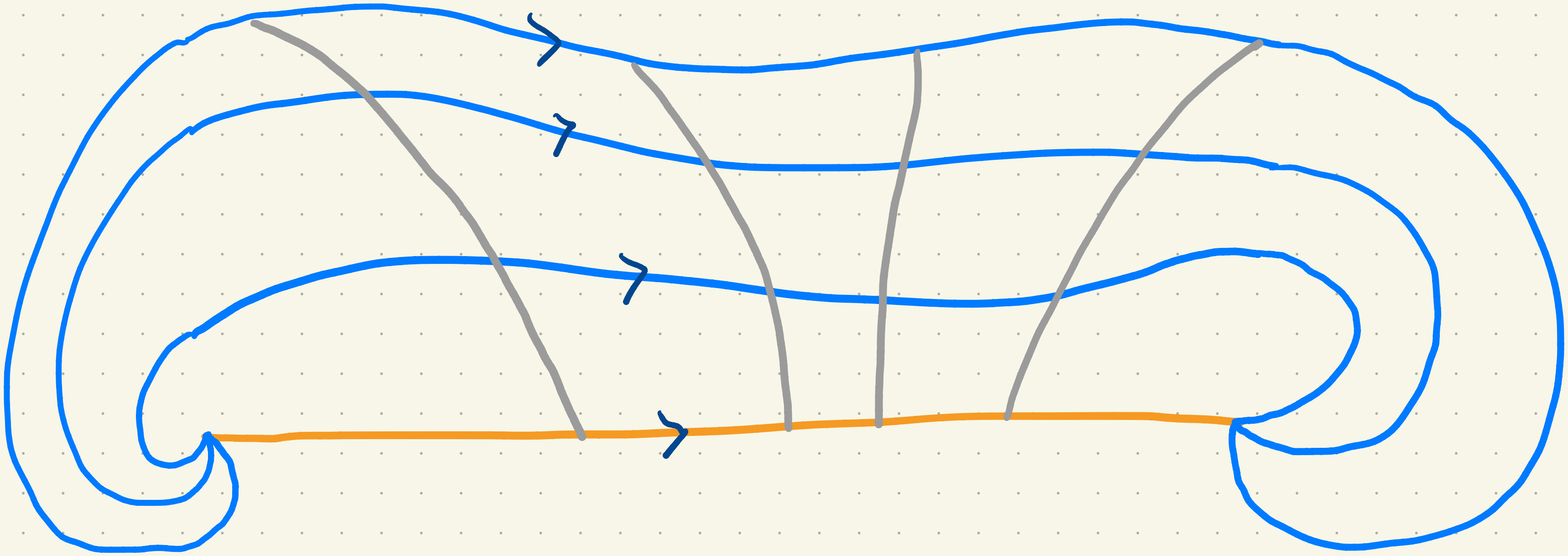
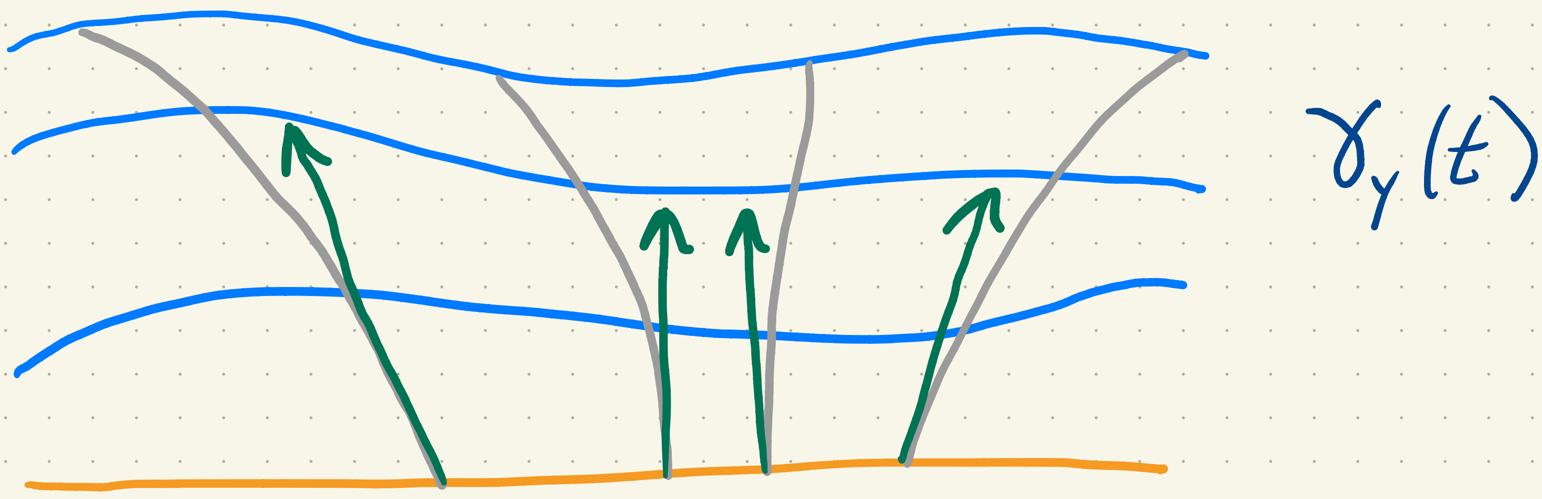


$\gamma_y(t)$

$$\frac{d}{dy}\Big|_{y=0} \int_{\gamma_y} \eta$$

Exterior Derivative of Covectors



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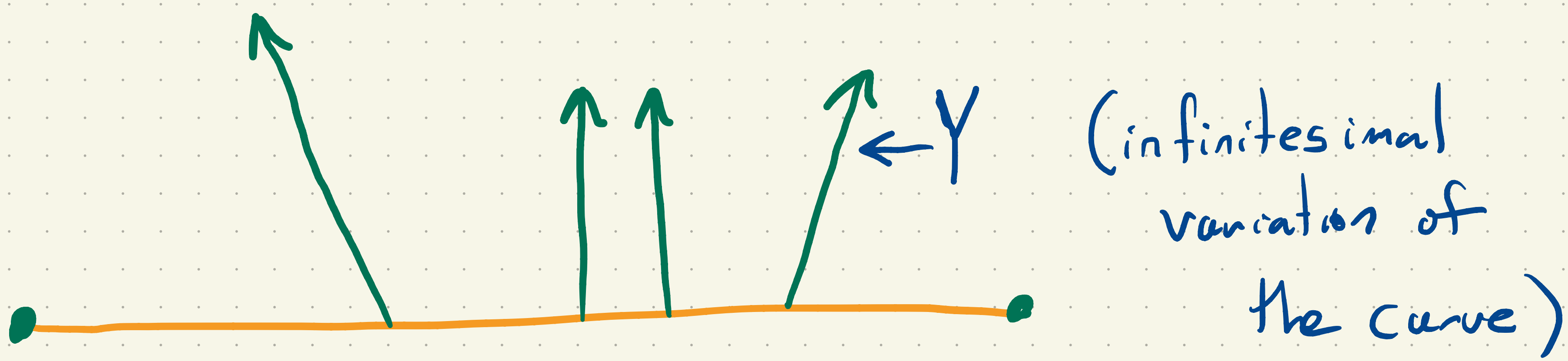

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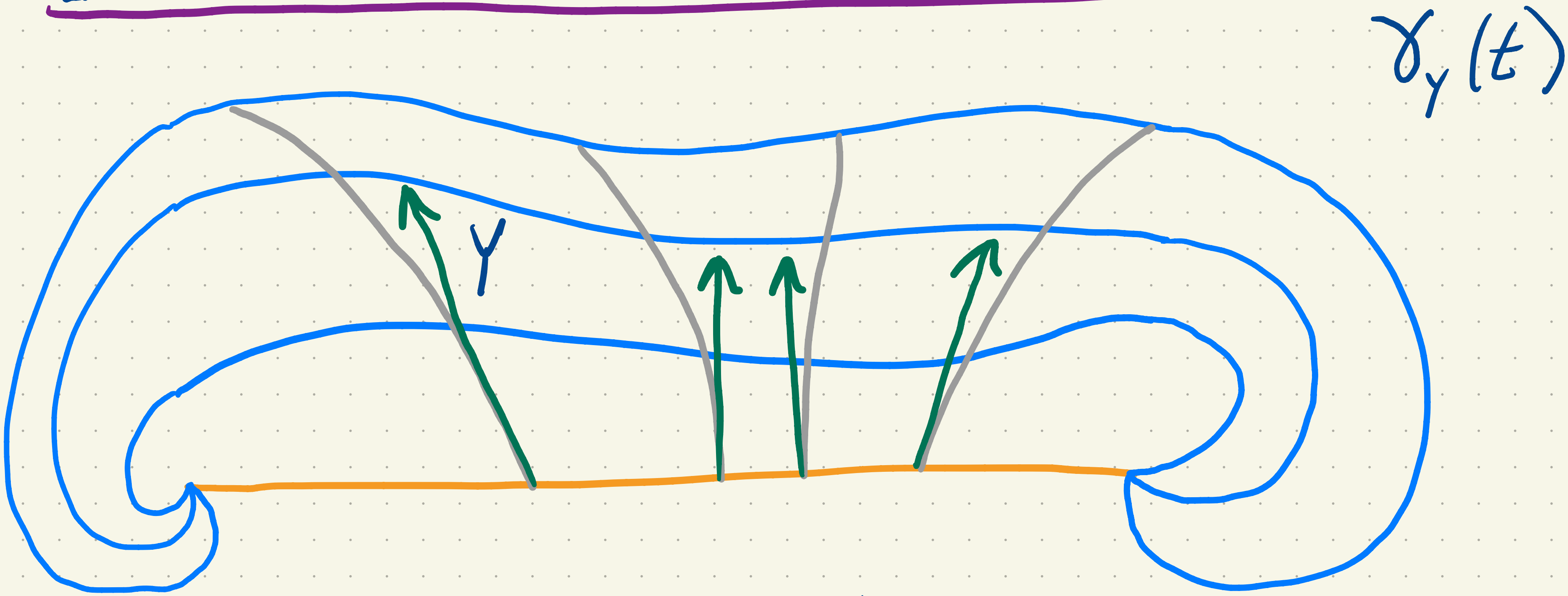
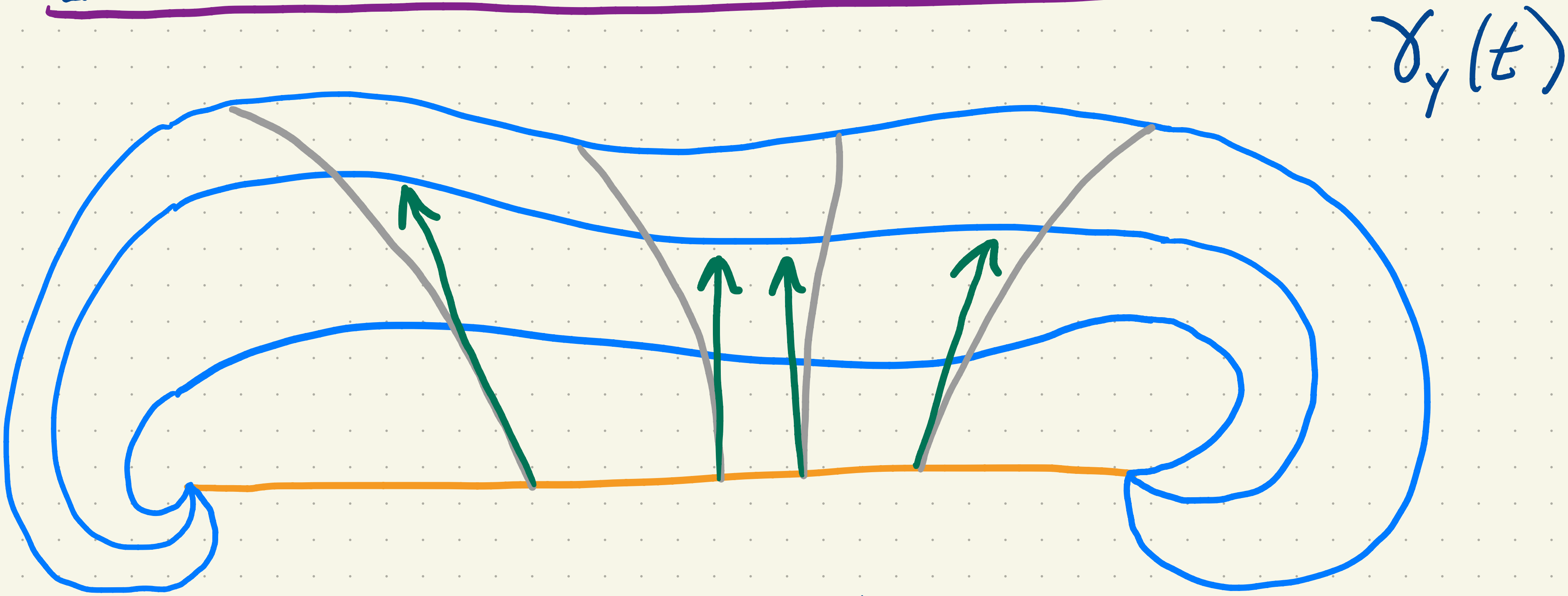


$$\int_\gamma d\eta[Y,\cdot] = \frac{d}{dy}\bigg|_{y=0} \int_{\gamma_y} \eta$$

$$Y \longrightarrow \int_\gamma d\eta[Y,\cdot]$$

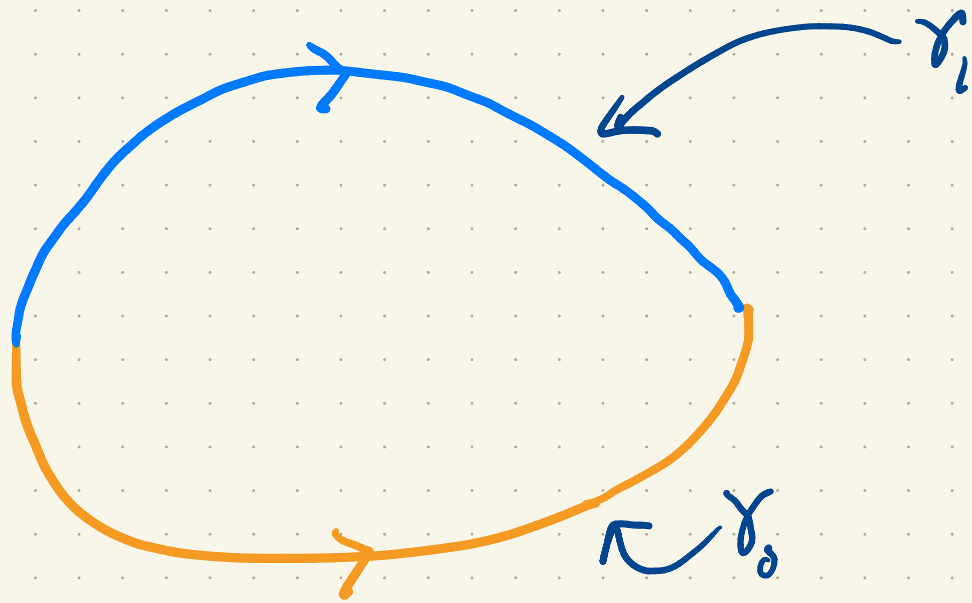
Exterior Derivative of Covectors



$$\int_\gamma d\eta[Y,\cdot] = \frac{d}{dy}\bigg|_{y=0} \int_{\gamma_y} \eta$$

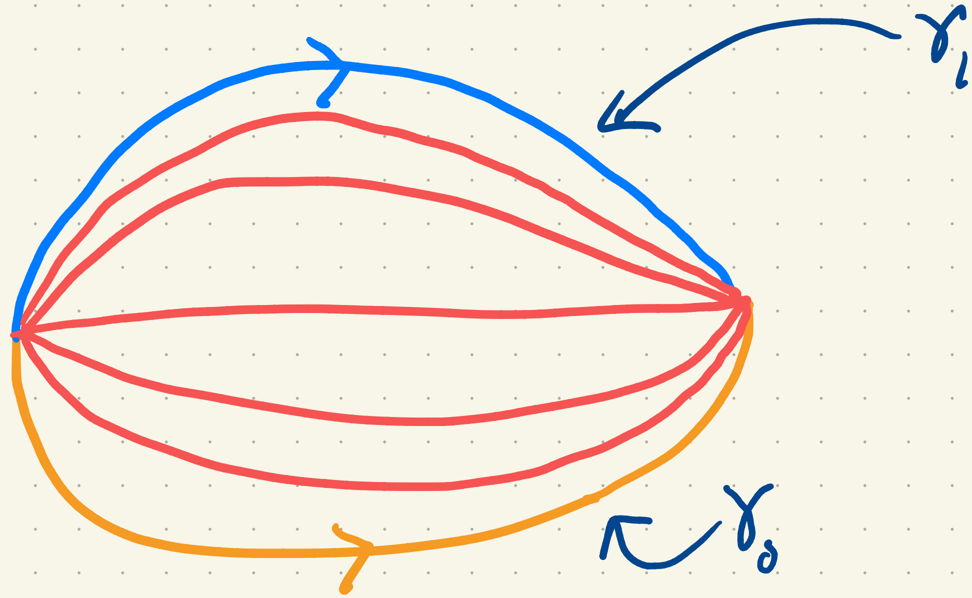
- If you make the infinitesimal variation Y of γ, how does $\int_\gamma \eta$ change?

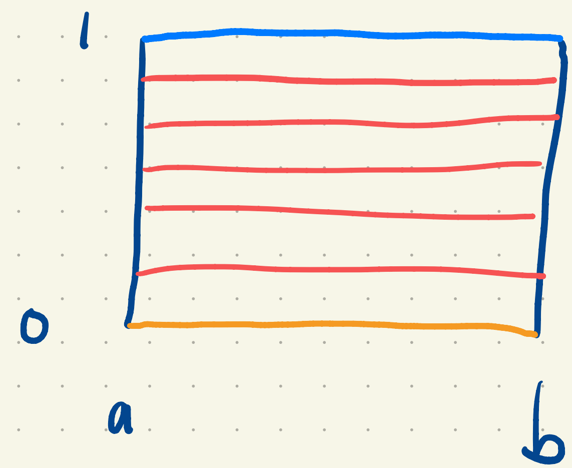
Stokes' Theorem (Preview)



$$\int_{\gamma_1} \eta - \int_{\gamma_0} \eta = \quad ?$$

Stokes' Theorem (Preview)





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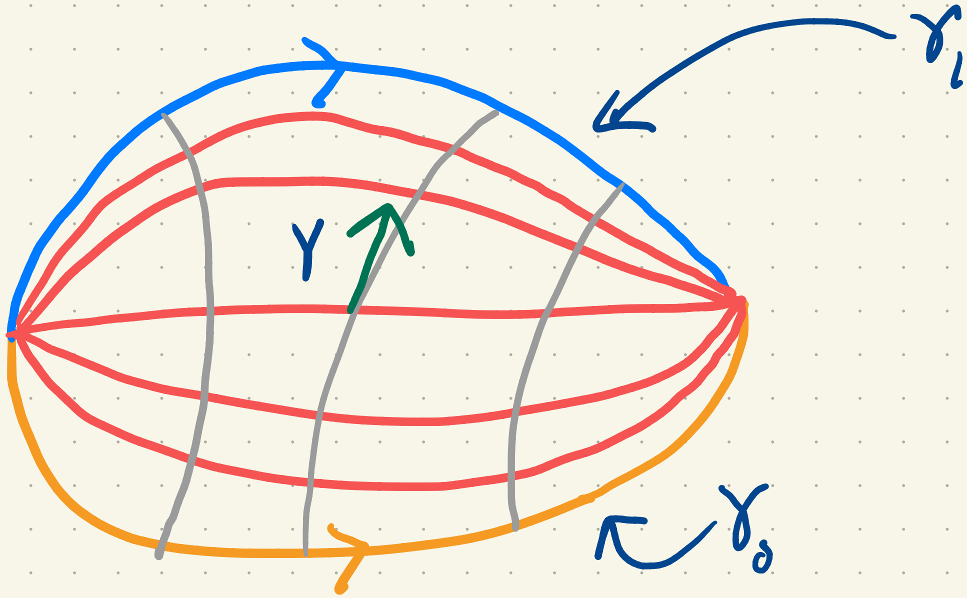

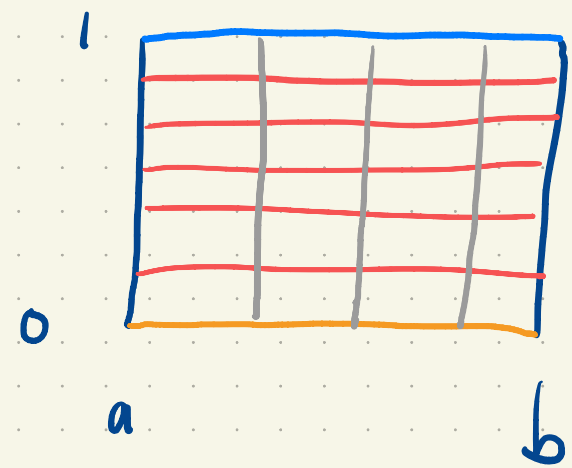

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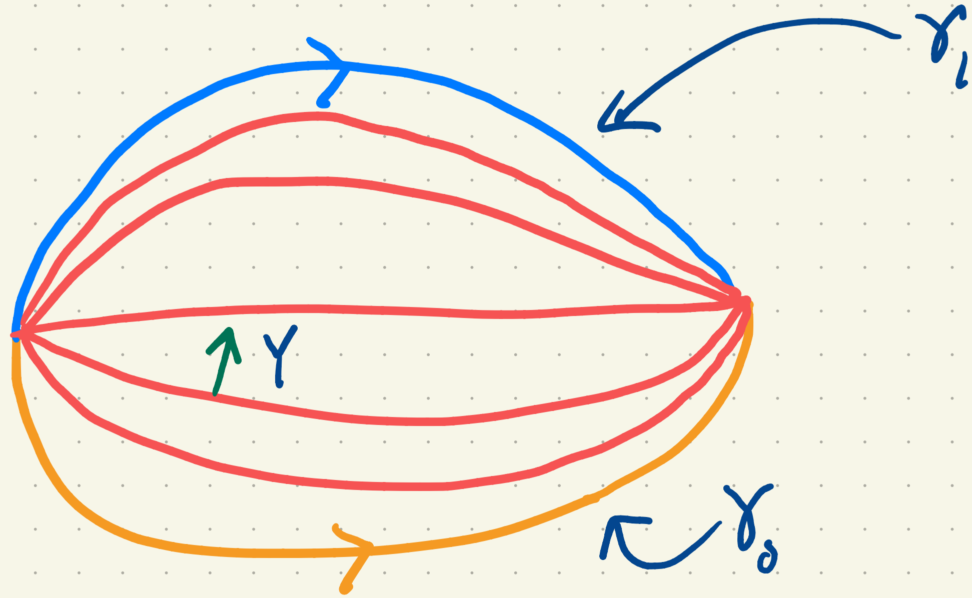
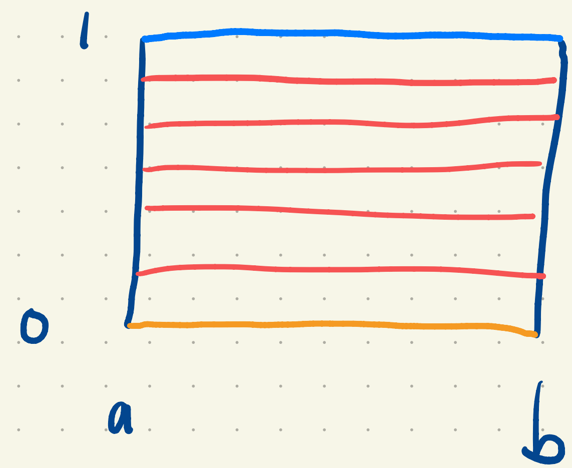
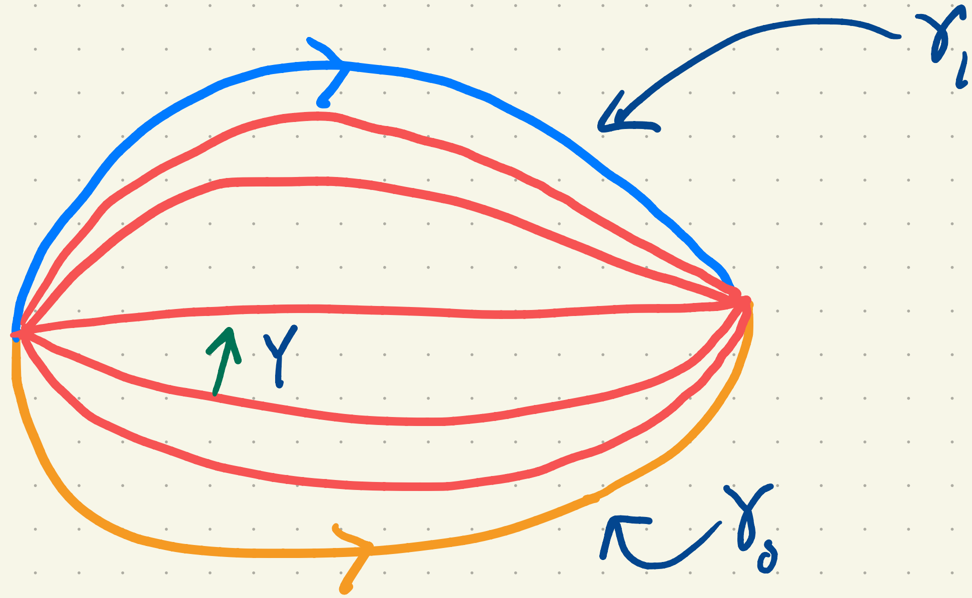
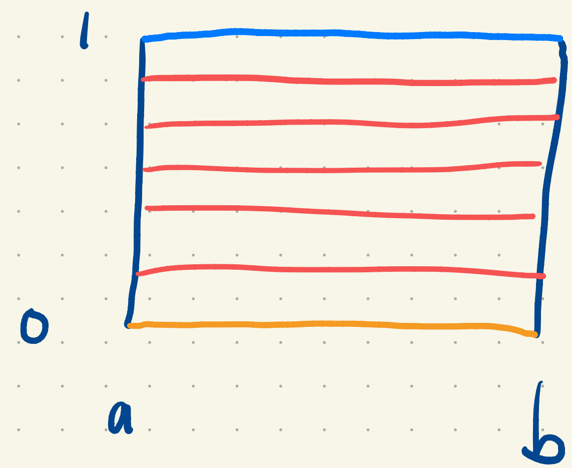




$$\int_{\gamma_1} \eta - \int_{\gamma_0} \eta = \int_0^1 \frac{d}{dy} \int_{\gamma_y} \eta \quad dy$$

Stokes' Theorem (Preview)





$$\int_{\gamma_1} \eta - \int_{\gamma_0} \eta = \int_0^1 \int_a^b d\eta(Y, \dot{\gamma}_y)\, dt\, dy$$

2-forms Eat Infinitesimal Planes

$$\eta[X] \rightsquigarrow \omega(X, Y)$$

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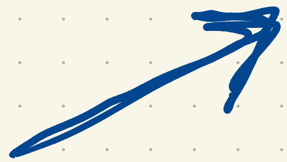

Rules: • ω is linear in each argument

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Rules:

- ω is linear in each argument
- $\omega(X,X) = 0$

2-forms Eat Infinitesimal Planes

$$\eta[X] \rightsquigarrow \omega(X, Y)$$

Rules:

- ω is linear in each argument
- $\omega(X, X) = 0$ $\qquad X+Y$
- $\omega(X, Y) = -\omega(Y, X)$

2-forms Eat Infinitesimal Planes

$$\eta[X] \rightsquigarrow \omega(X, Y)$$

Rules:

- ω is linear in each argument
- $\omega(X, X) = 0$
- $\omega(X, Y) = -\omega(Y, X)$

$$\boxed{d\eta \text{ is a 2-form if } \eta \text{ is a 1-form}}$$

2-forms Eat Infinitesimal Planes

$$\eta[X] \rightsquigarrow \omega(X, Y)$$

How to make a two form from covectors η, μ:

$$(\eta \wedge \mu)(X, Y) = \eta[X]\,\mu[Y] - \eta[Y]\,\mu[X]$$

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$$\eta \wedge \mu = -\mu \wedge \eta$$

2-forms Eat Infinitesimal Planes

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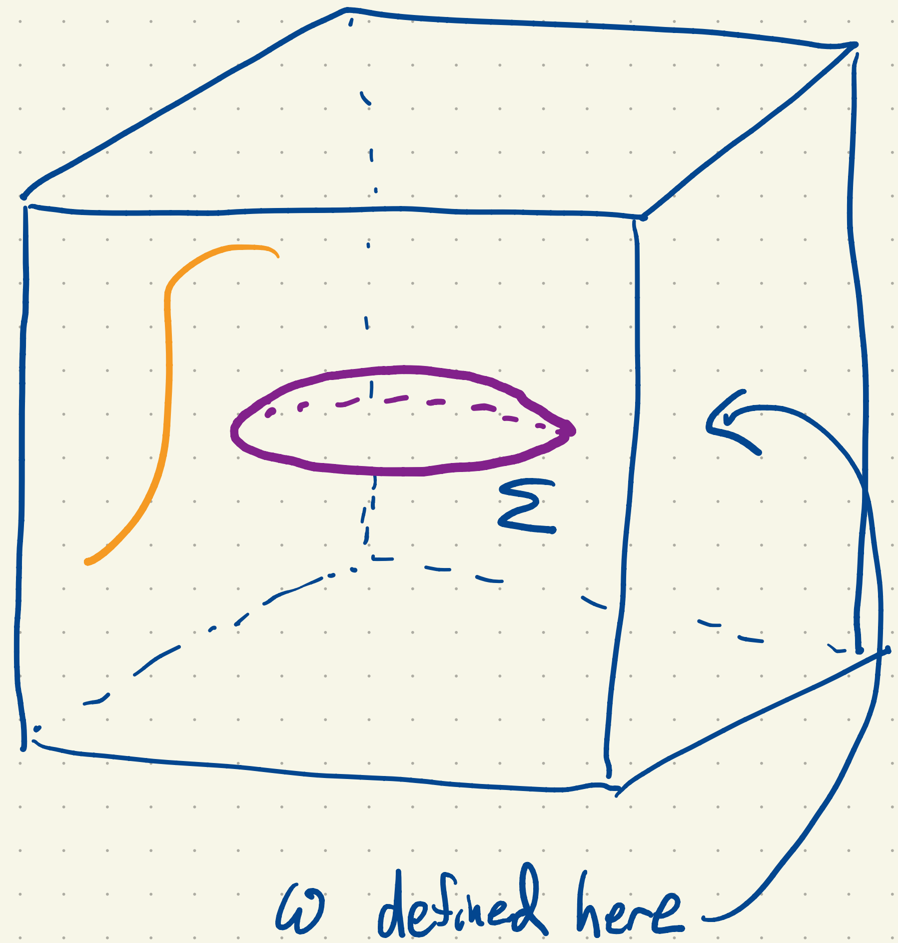
How to make a two form from du, dv:

$$du \wedge dv = -dv \wedge du$$

Fields of 2-forms Eat Surfaces



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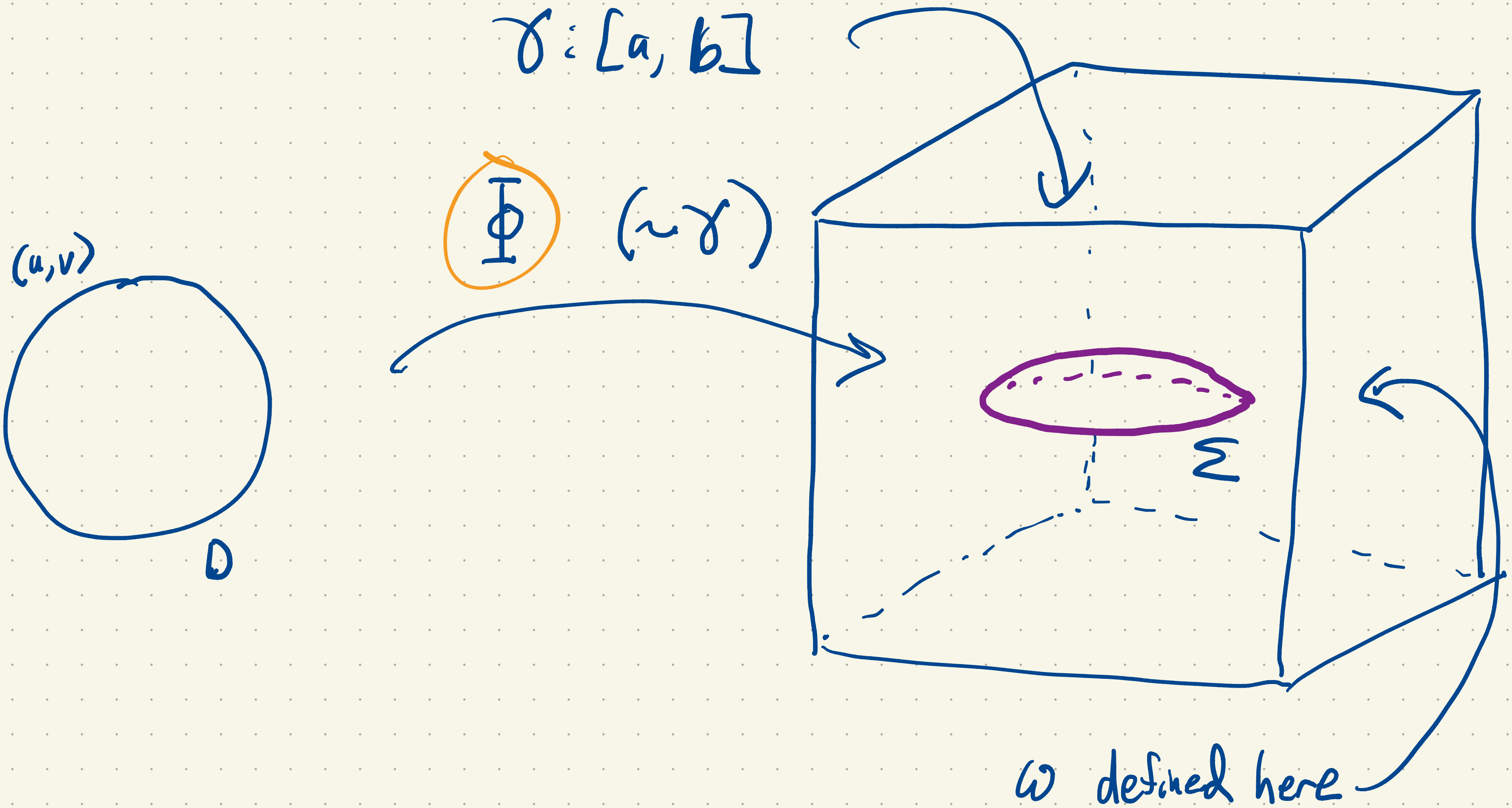

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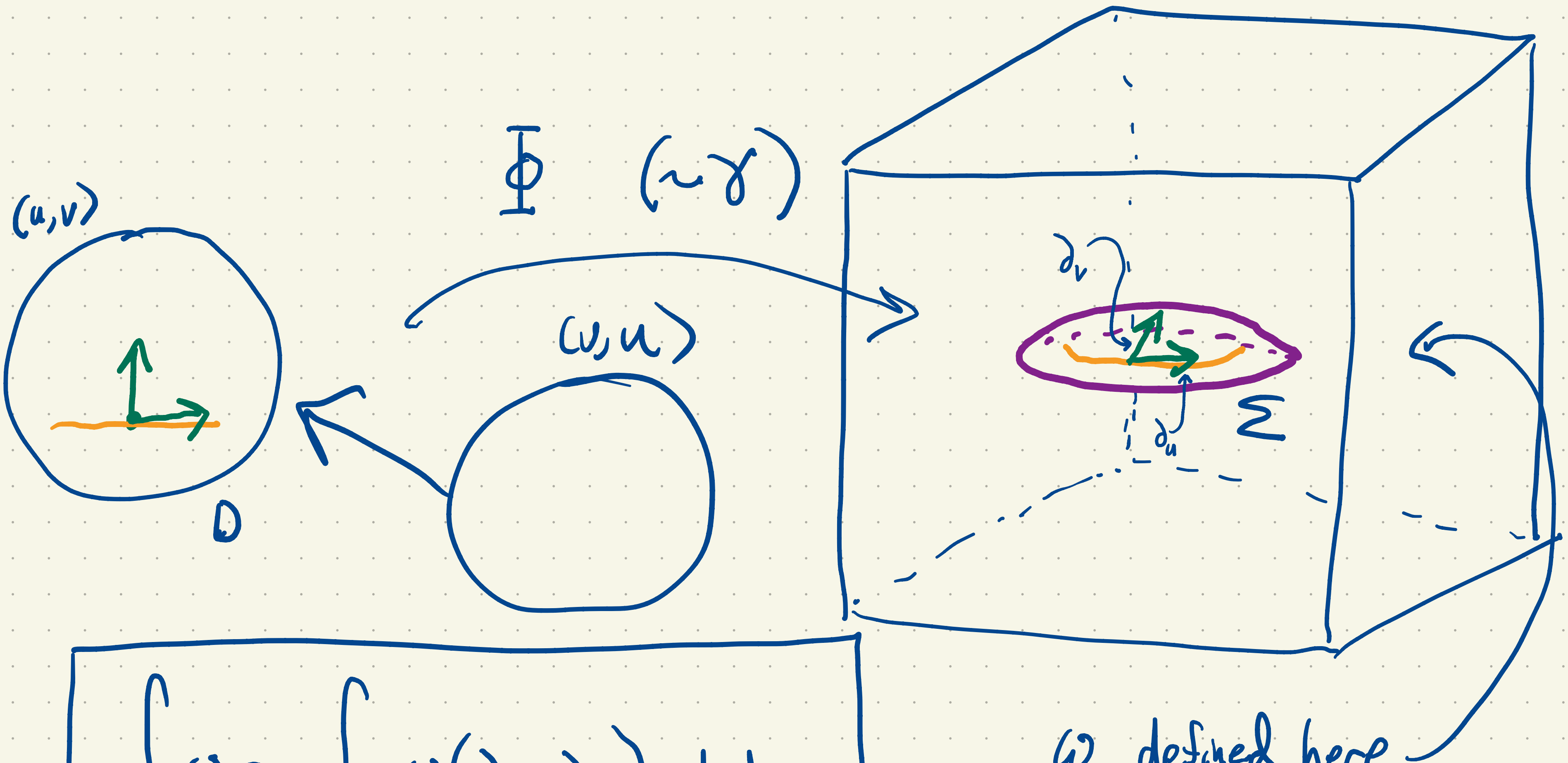
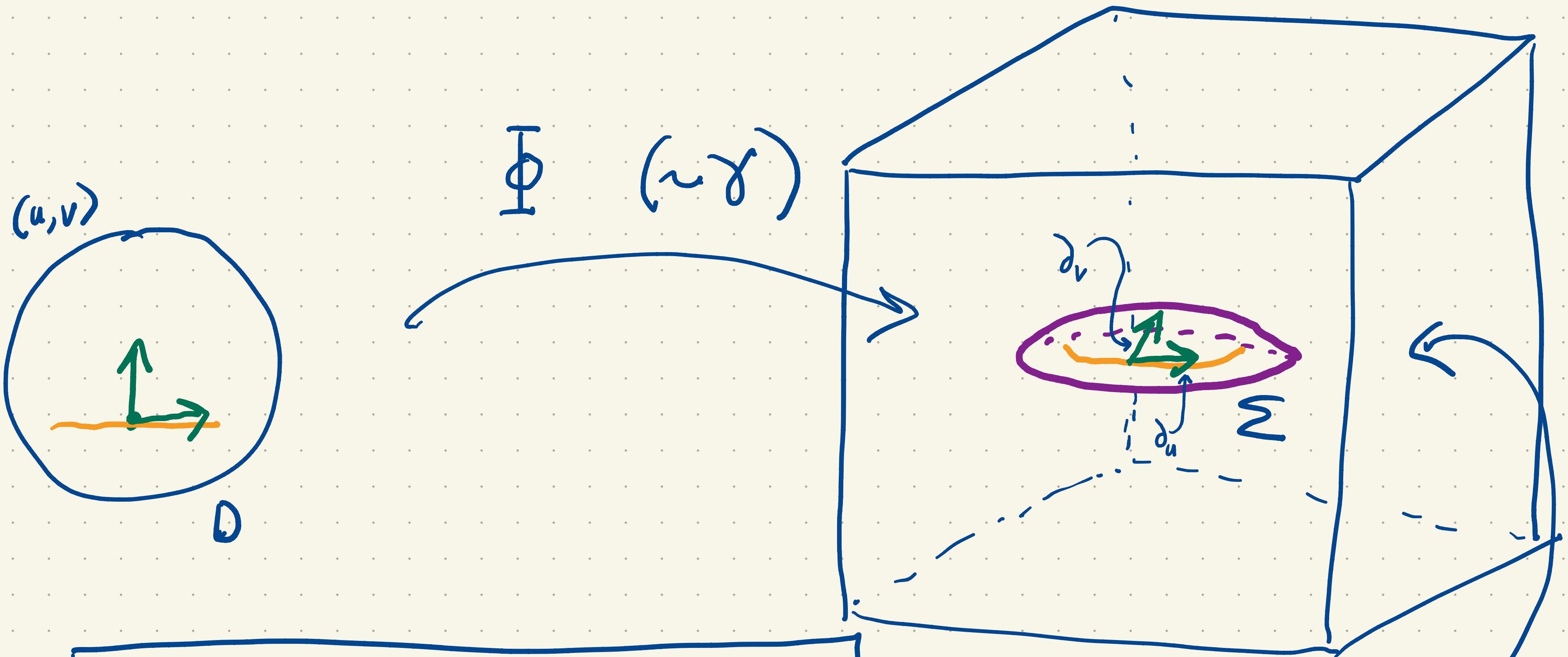


$$\int_\Sigma \omega = \int_D \omega(\partial_u, \partial_v)\, du\, dv$$

Fields of 2-forms Eat Surfaces

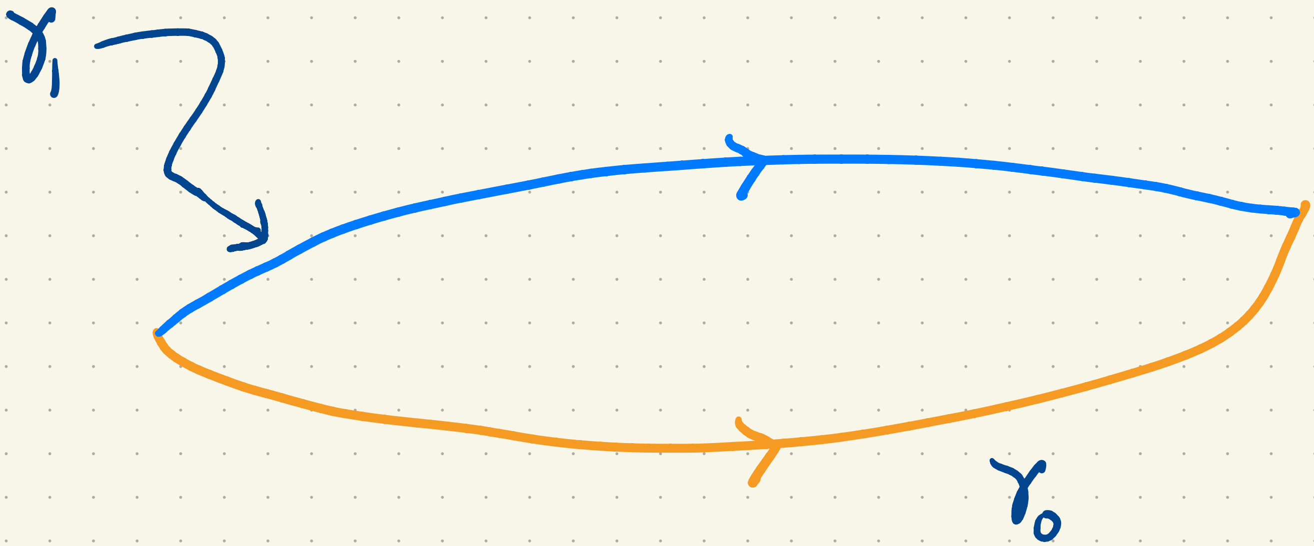


$$\boxed{\int_\Sigma \omega = \int_D \omega(\partial_u, \partial_v)\, du\, dv}$$

ω defined here

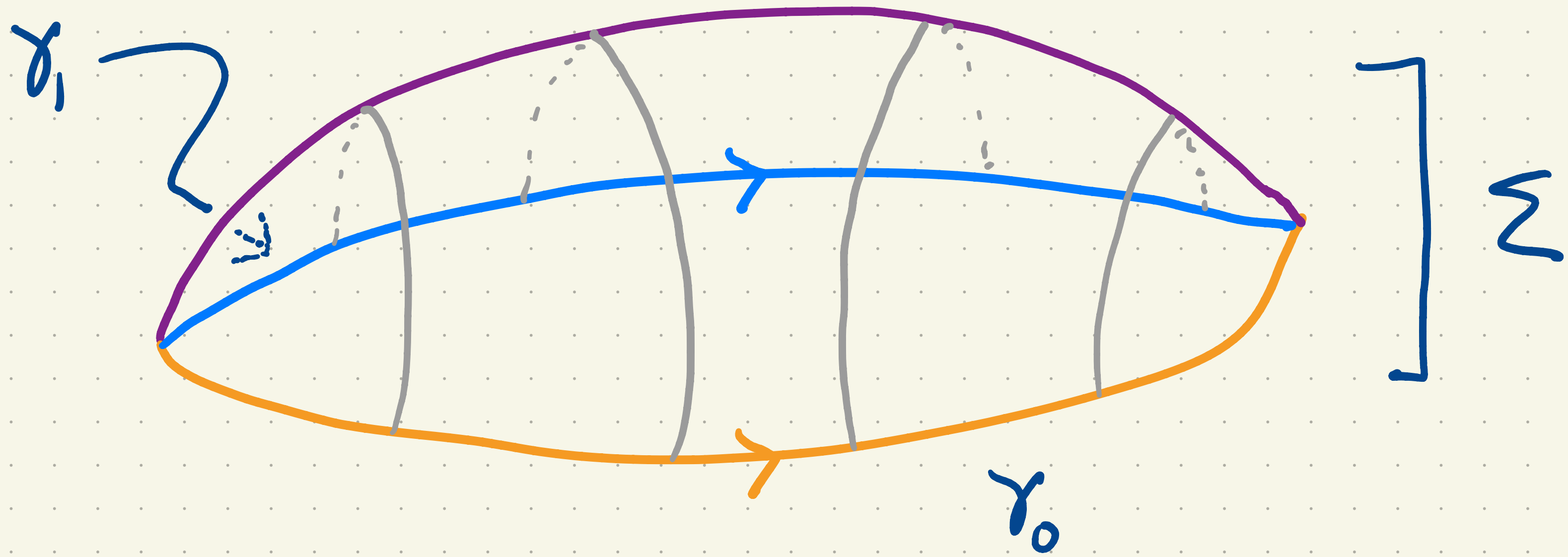
(orientation)

Stokes' Theorem



$$\int_{\gamma_1} \eta - \int_{\gamma_0} \eta$$

Stokes' Theorem



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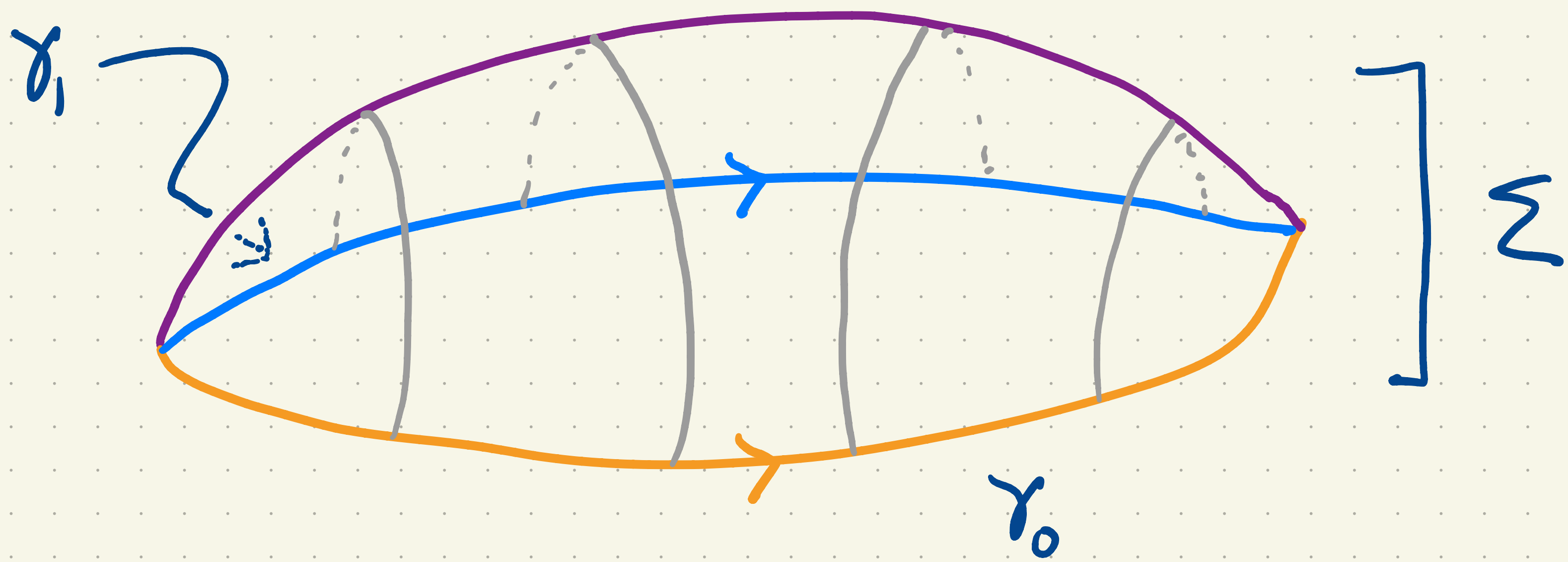
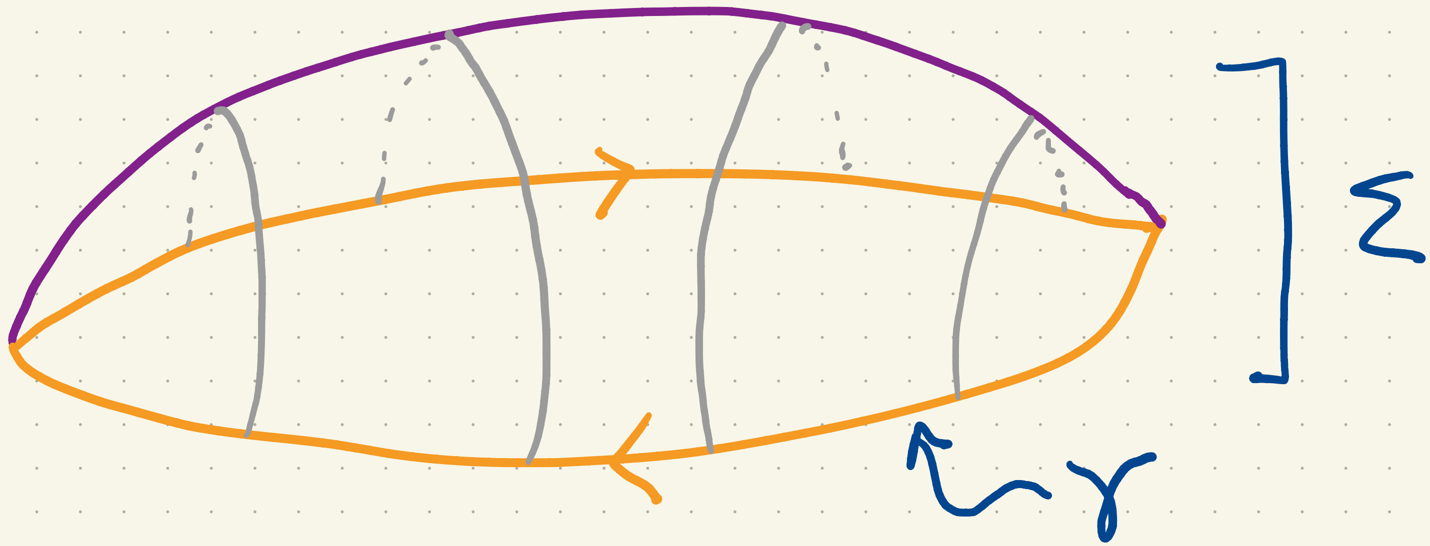


$$\int_{\Sigma} d\eta = \int_{\gamma_1} \eta - \int_{\gamma_0} \eta$$

Stokes' Theorem



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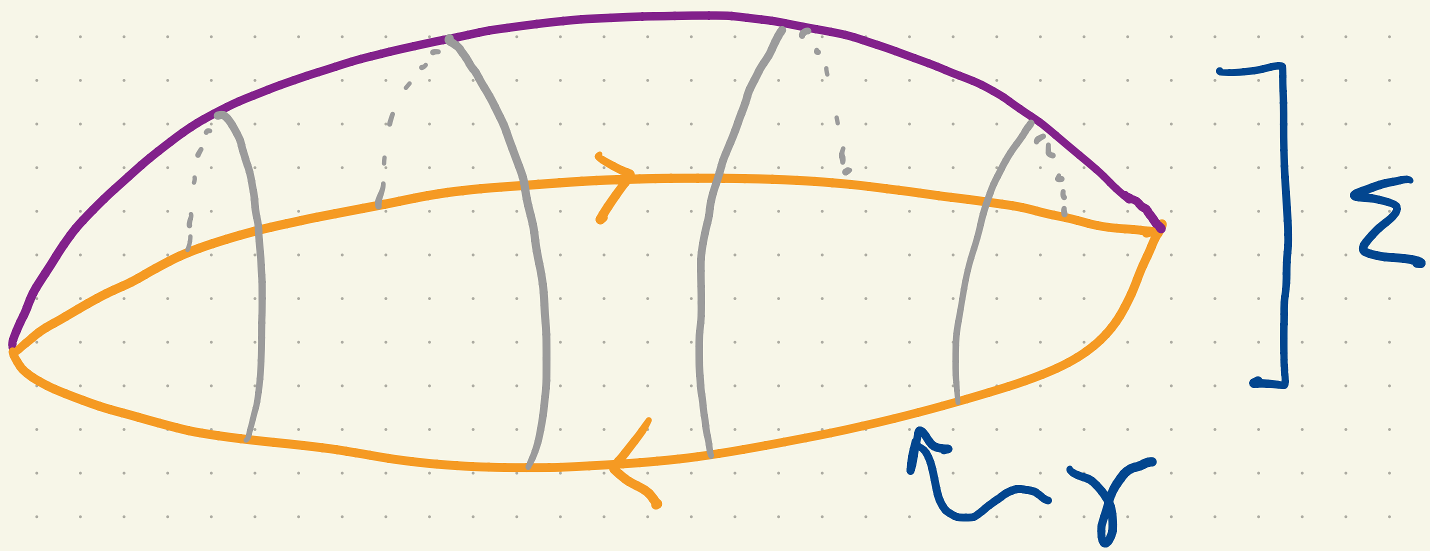

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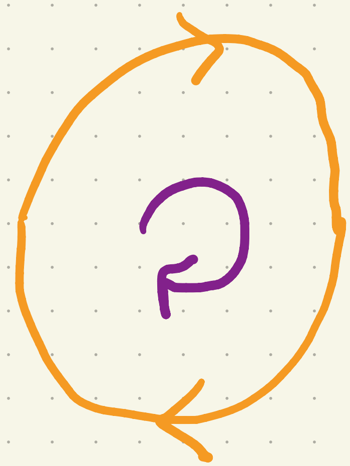

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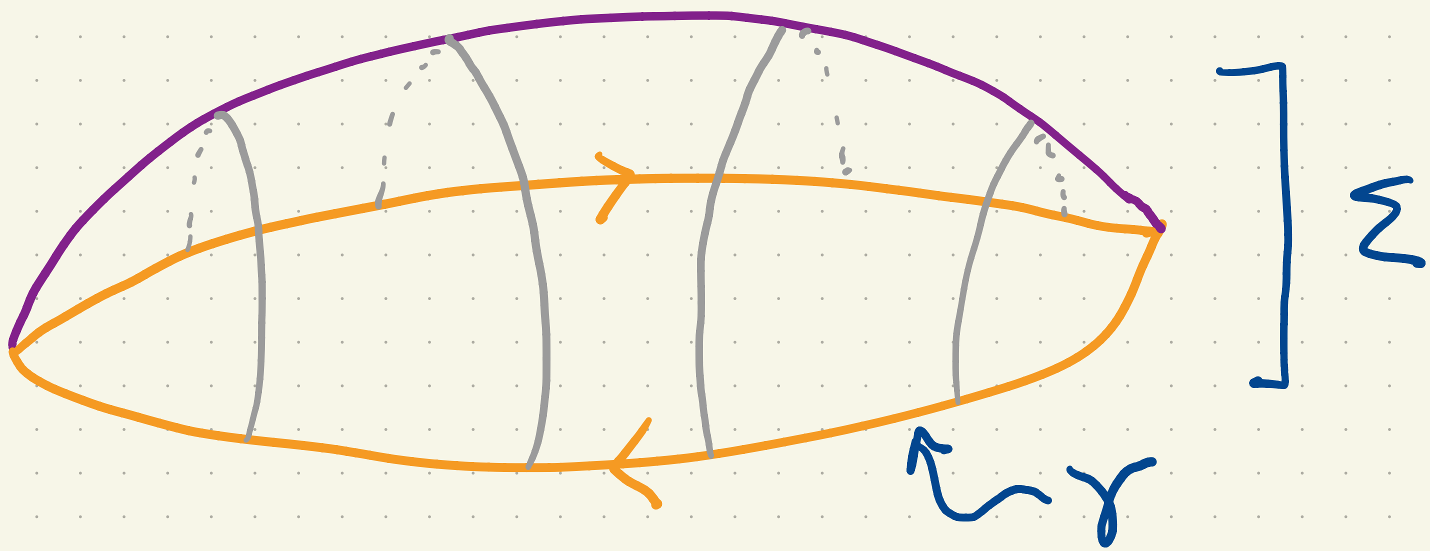

$$\int_\Sigma d\eta = \int_\gamma \eta$$

Yes, This Generalizes

- function = 0-form $\xrightarrow{d}$ 1-form = covector
- 1-form $\xrightarrow{d}$ 2-form
- 2-form $\xrightarrow{d}$ 3-form

$$\vdots$$

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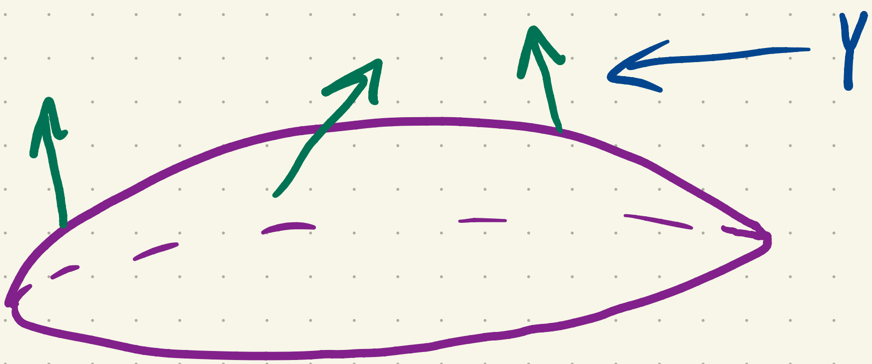

$$\int_{\Sigma} d\omega[Y, \cdot, \cdot]$$

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$$\int_{\Omega^n} d\alpha = \int_{\partial\Omega^{n-1}} \alpha$$

How to Compute

- $d(df) = 0$

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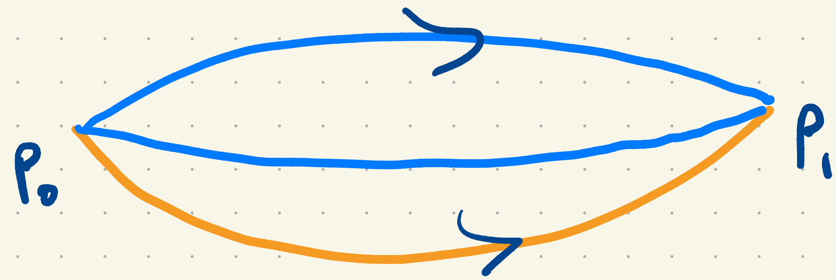

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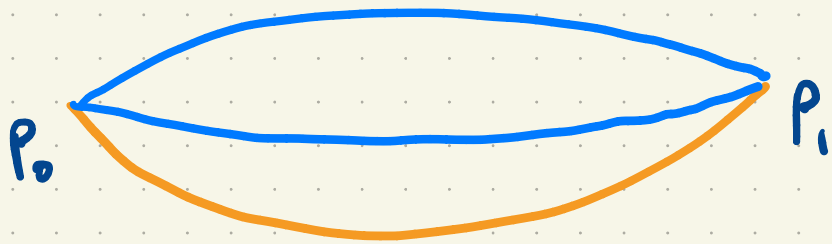

$$\int_{\gamma_y} df = f(p_1) - f(p_0)$$

1-form

- $$d(f\eta) = df \wedge \eta + f\, d\eta$$

function

How to Compute

- $d(df) = 0$
- $d(f\eta) = df \wedge \eta + f\, d\eta$
- $d\big(A(x,y)\,dx + B(x,y)\,dy\big)$

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$$(\partial_x A\, dx + \partial_y A\, dy) \wedge dx$$

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$$= \partial_y A \, dy \wedge dx + \partial_x B \, dx \wedge dy$$

$$= (\partial_x B - \partial_y A)\, dx \wedge dy$$

How to Compute

- $d(A\,dx + B\,dy) = (\partial_x B - \partial_y A)\, dx \wedge dy$
- $d\big(A(x,y,z)\,dx + B(x,y,z)\,dy + C(x,y,z)\,dz\big)$

$$= (\partial_x B - \partial_y A)\, dx \wedge dy +$$
$$(\partial_y C - \partial_z B)\, dy \wedge dz +$$
$$(\partial_z A - \partial_x C)\, dz \wedge dx$$

How to Compute

- $d(\phi\, dt - A_x\, dx - A_y\, dy - A_z\, dz)$

  $\left[\phi = \phi(t, x, y, z), \text{ etc} \right]$

How to Compute

- $d(\phi\, dt - A_x\, dx - A_y\, dy - A_z\, dz)$

$$
\begin{aligned}
&= \left[(\partial_x \phi + \dot{A}_x)\, dx + (\partial_y \phi + \dot{A}_y)\, dy + (\partial_z \phi + \dot{A}_z)\, dz \right] \wedge dt \\
&\quad - (\partial_x A_y - \partial_y A_x)\, dx \wedge dy - (\partial_y A_z - \partial_z A_y)\, dy \wedge dz \\
&\quad - (\partial_z A_x - \partial_x A_z)\, dz \wedge dx
\end{aligned}
$$

Yes, This is Foreshadowing

$$\vec{E} = \vec{\nabla}\phi + \dot{\vec{A}}$$
$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\left[\left(\partial_x \phi + \dot{A}_x\right) dx + \left(\partial_y \phi + \dot{A}_y\right) dy + \left(\partial_z \phi + \hat{A}_z\right) dz\right] \wedge dt$$

$$-\left(\partial_x A_y - \partial_y A_x\right) dx \wedge dy - \left(\partial_y A_z - \partial_z A_y\right) dy \wedge dz$$

$$-\left(\partial_z A_x - \partial_x A_z\right) dz \wedge dx$$

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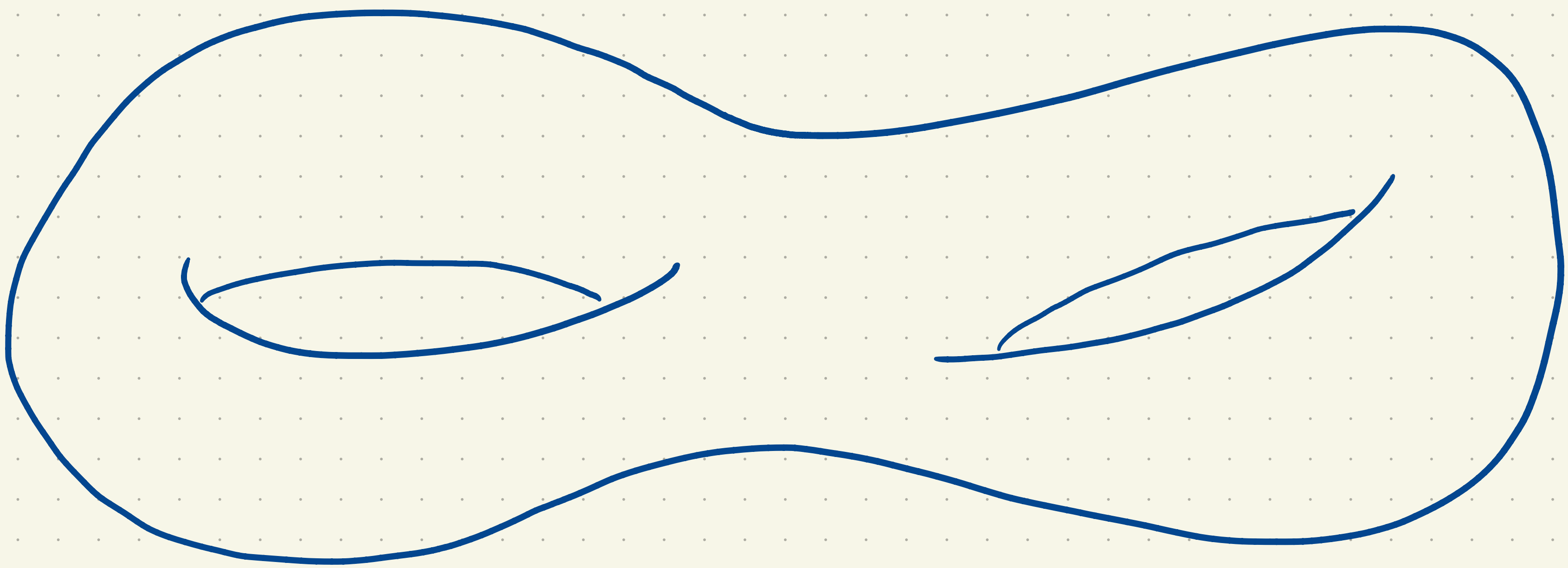
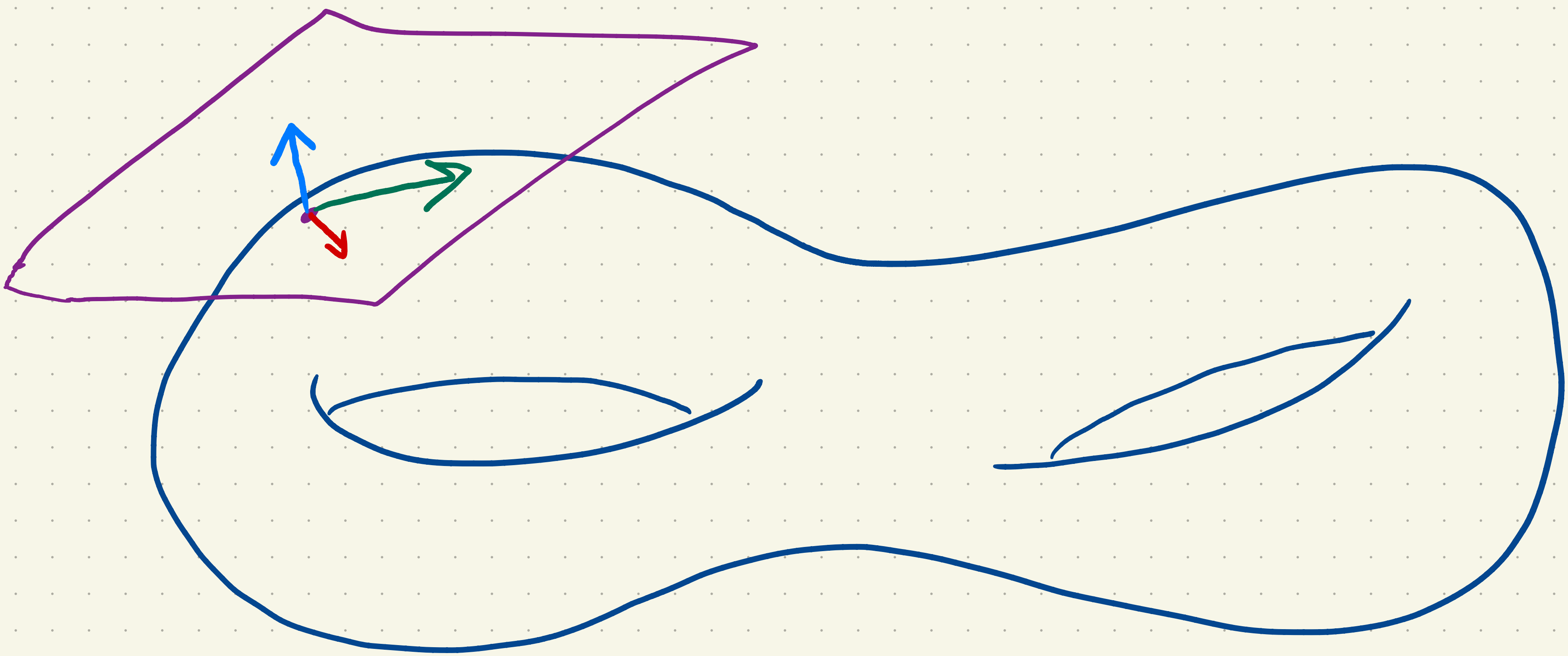
- This is vaguely intriguing, but not cool yet.
- The universe posesses a local symmetry unrelated to its shape.
- You can drag symmetry with you via a connection.
- Once you make a choice (akin to a choice of coordinates) you can represent the connection as a covector $\mathcal{A}$.
- $d\mathcal{A}$ encodes geometric information as well as $\vec{E}$ and $\vec{B}$.

Local Symmetry via Geometry



Local Symmetry via Geometry



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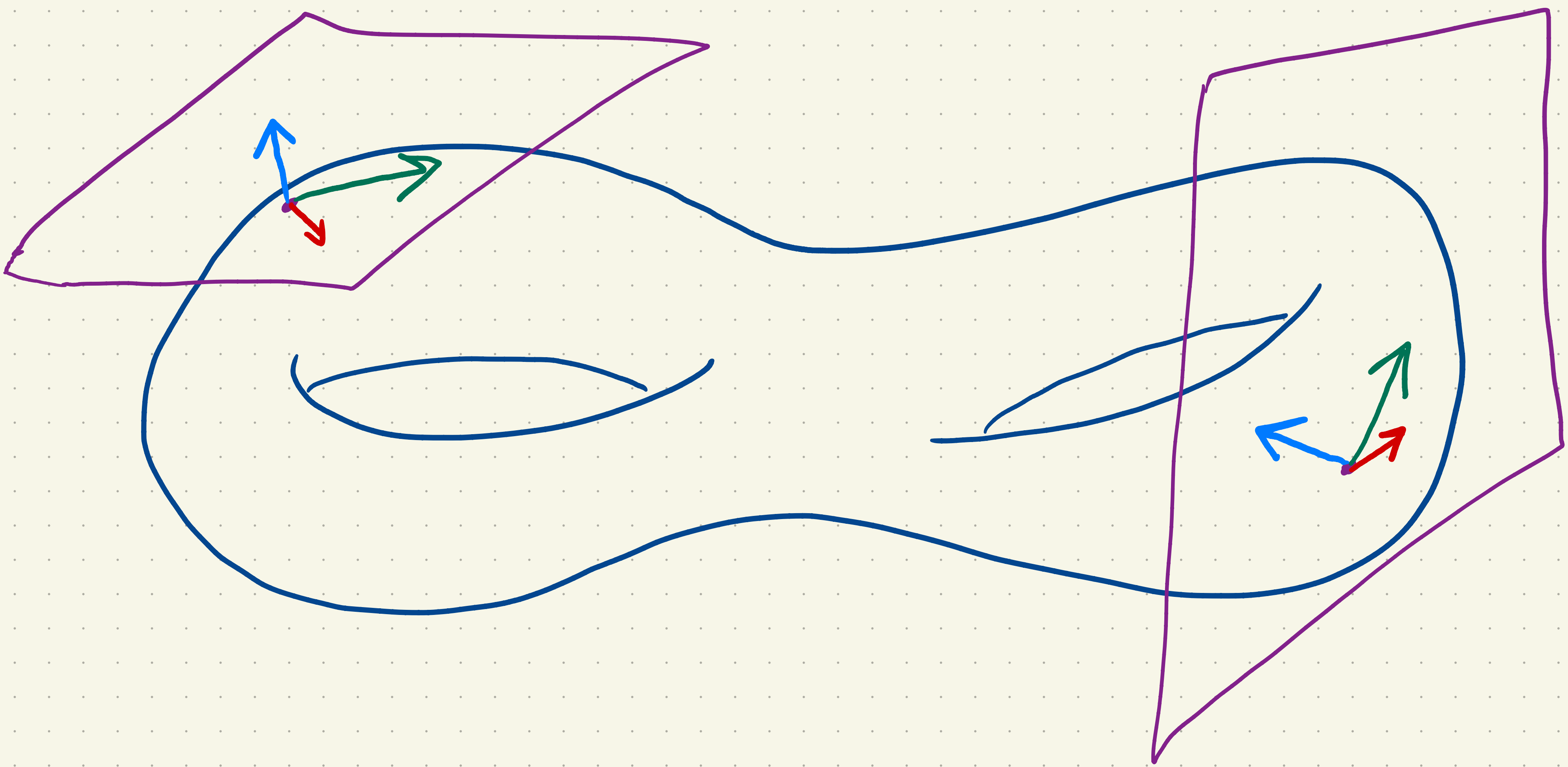
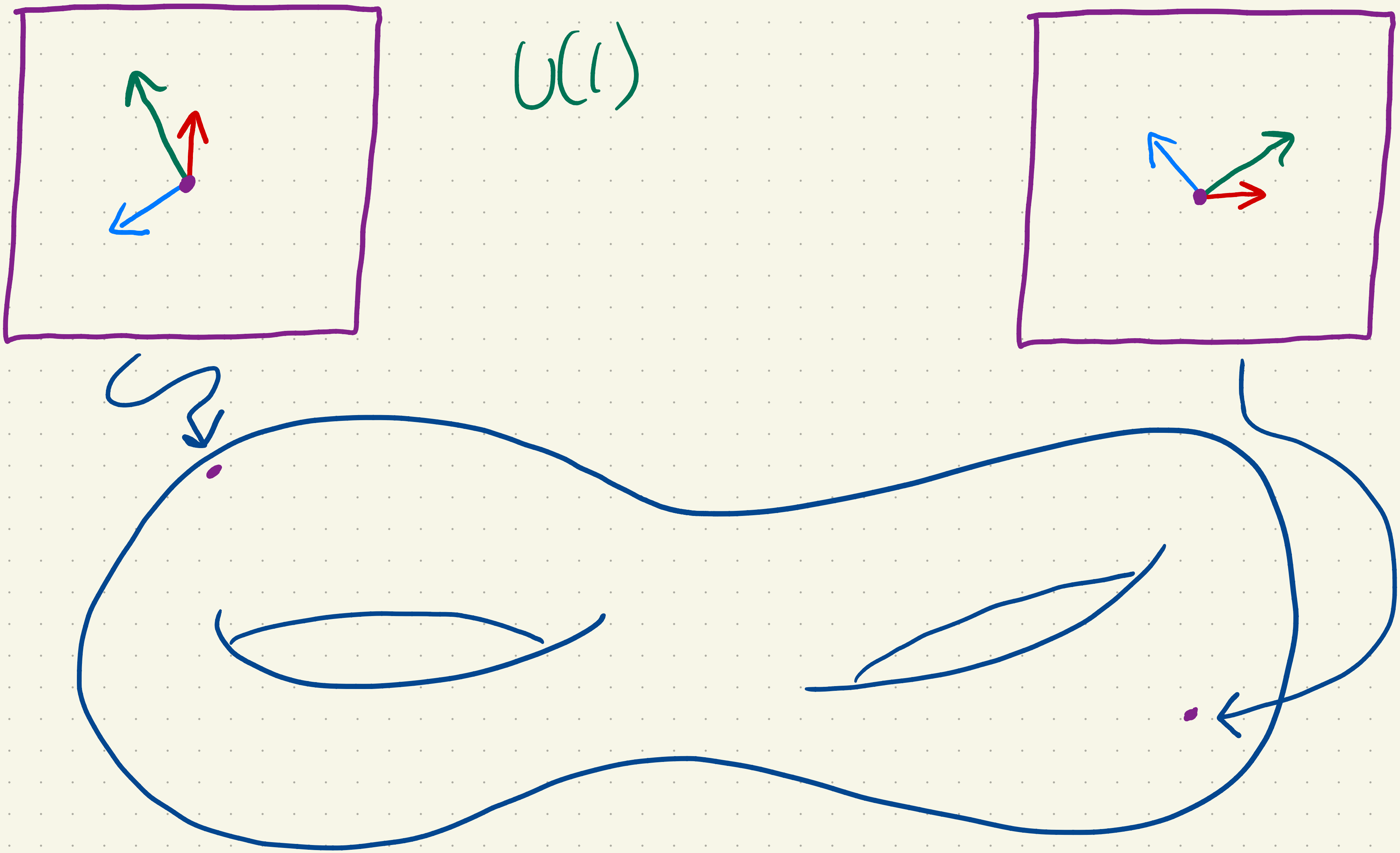
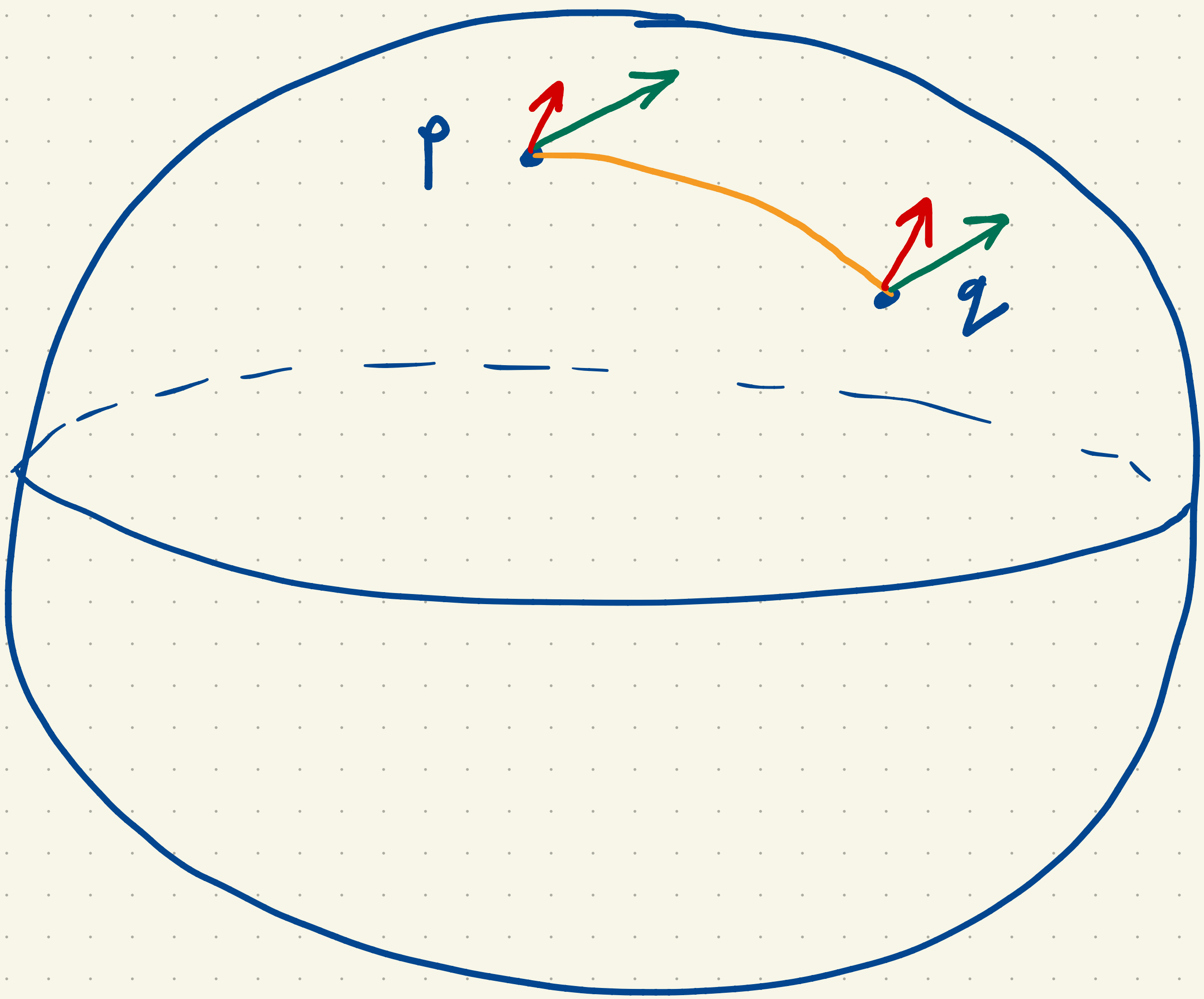


Local Symmetry



Symmetry Dragging



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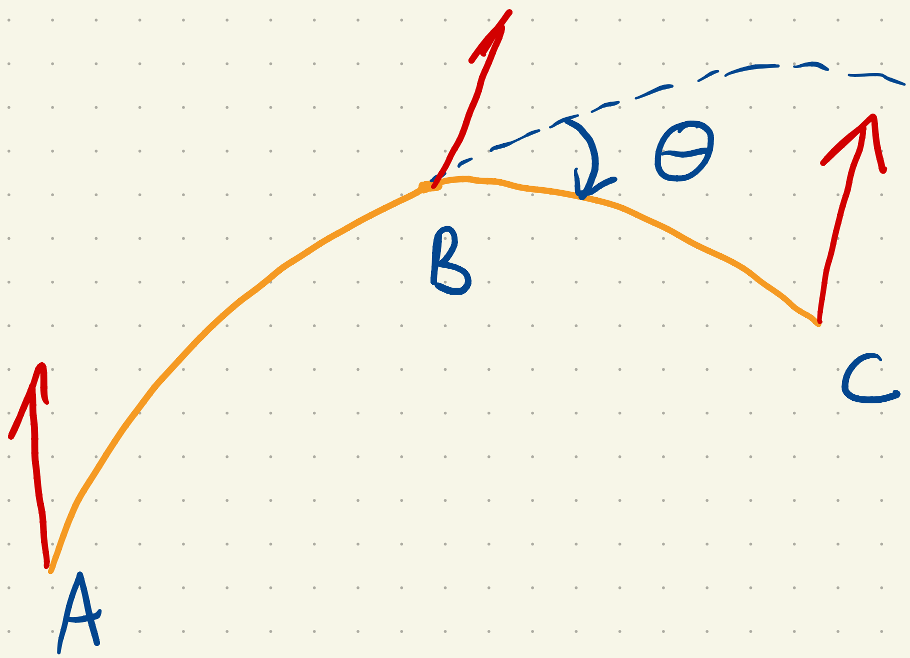

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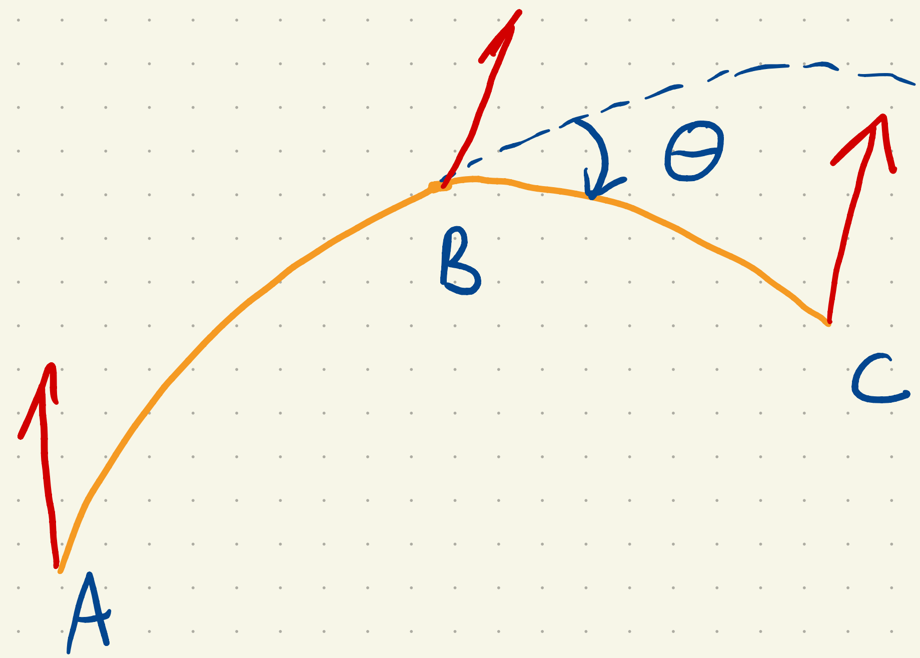
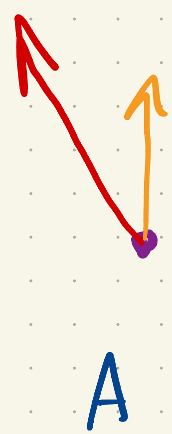
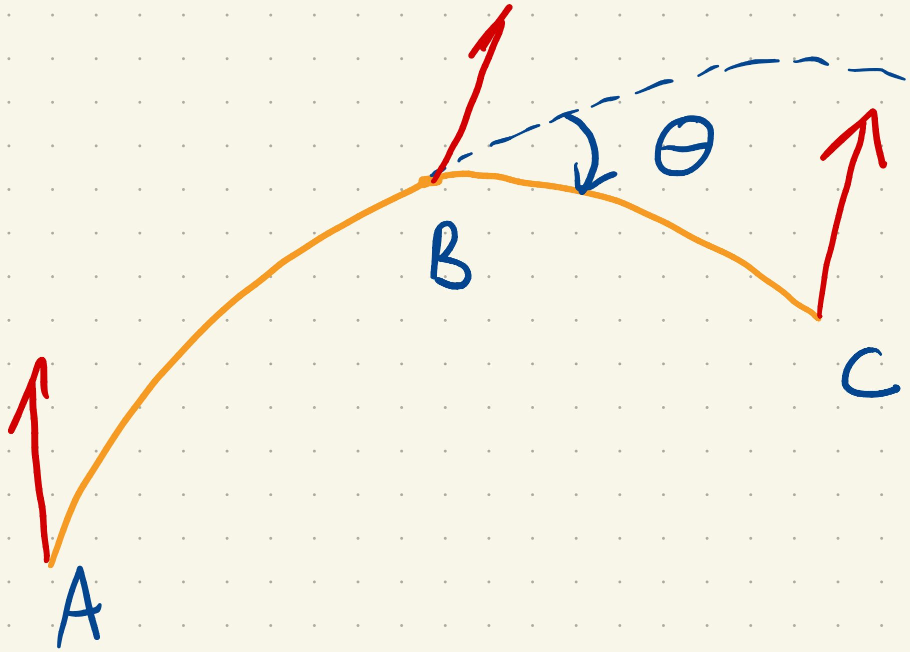
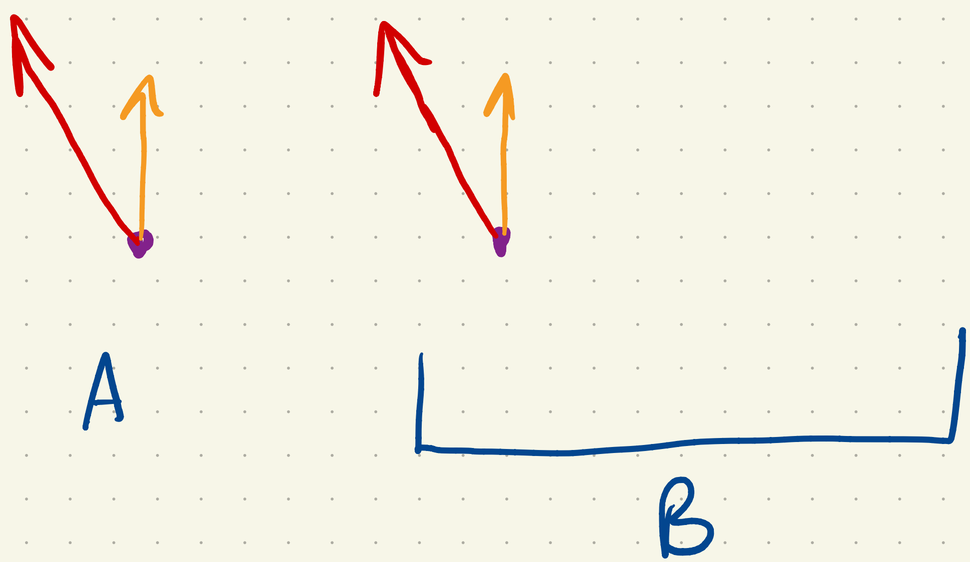


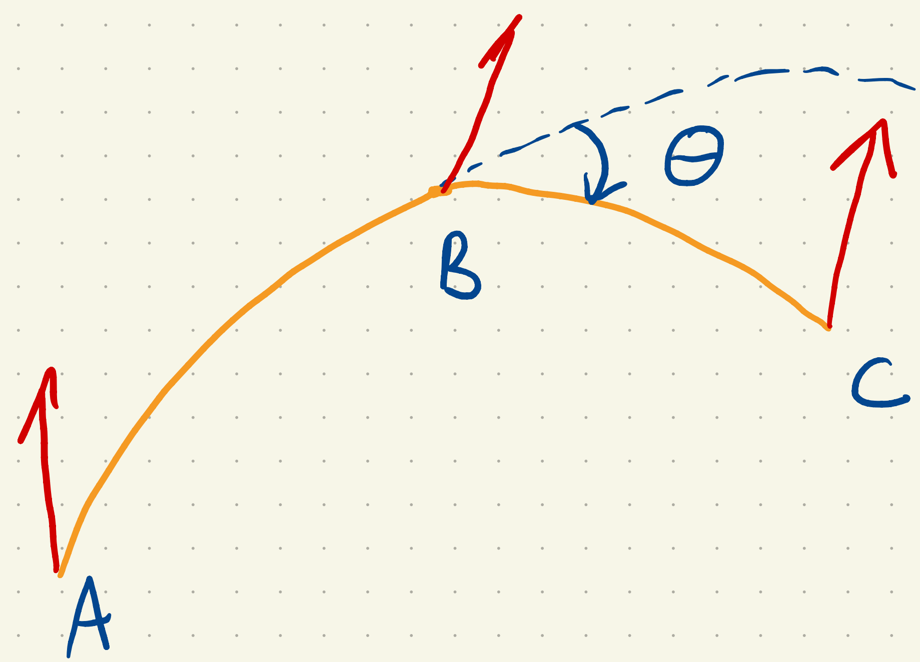


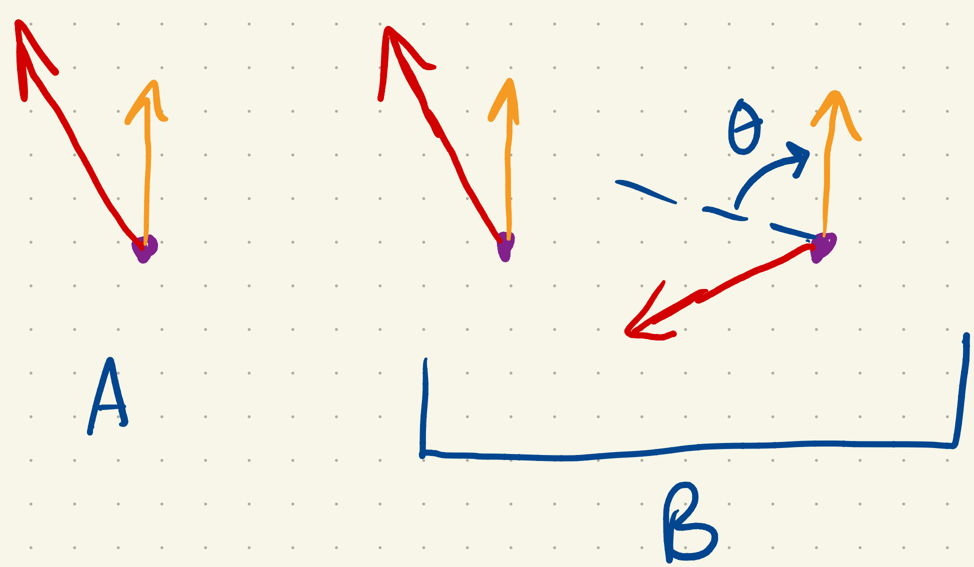
Symmetry Dragging





Symmetry Dragging





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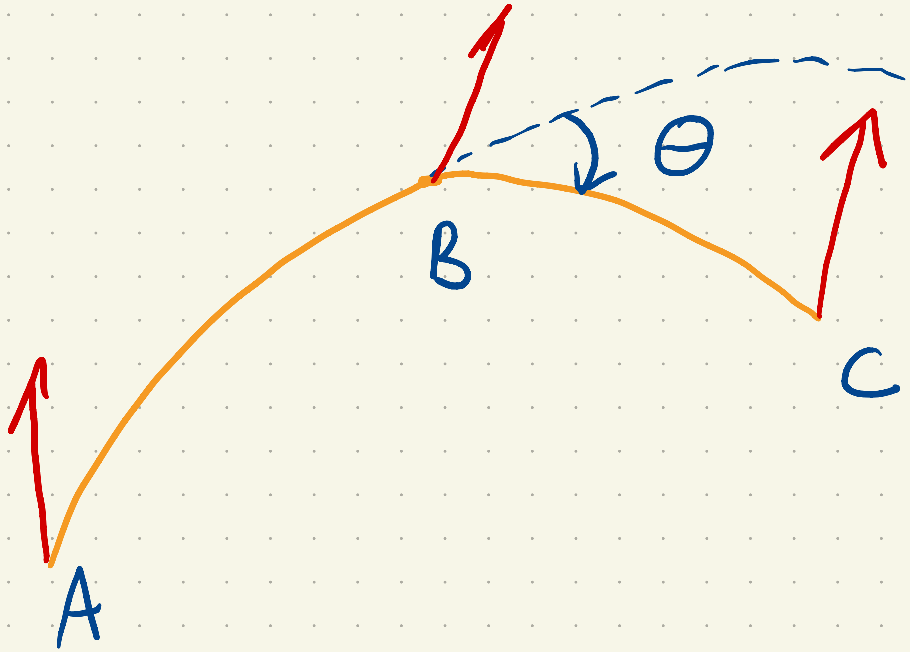
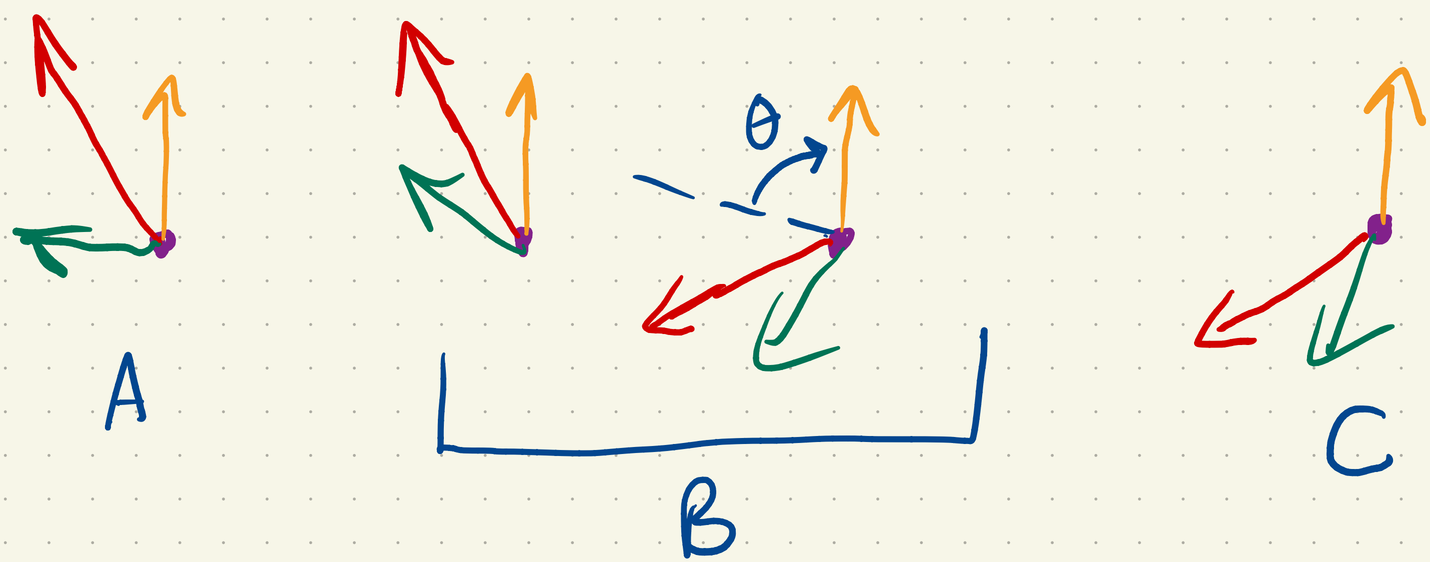
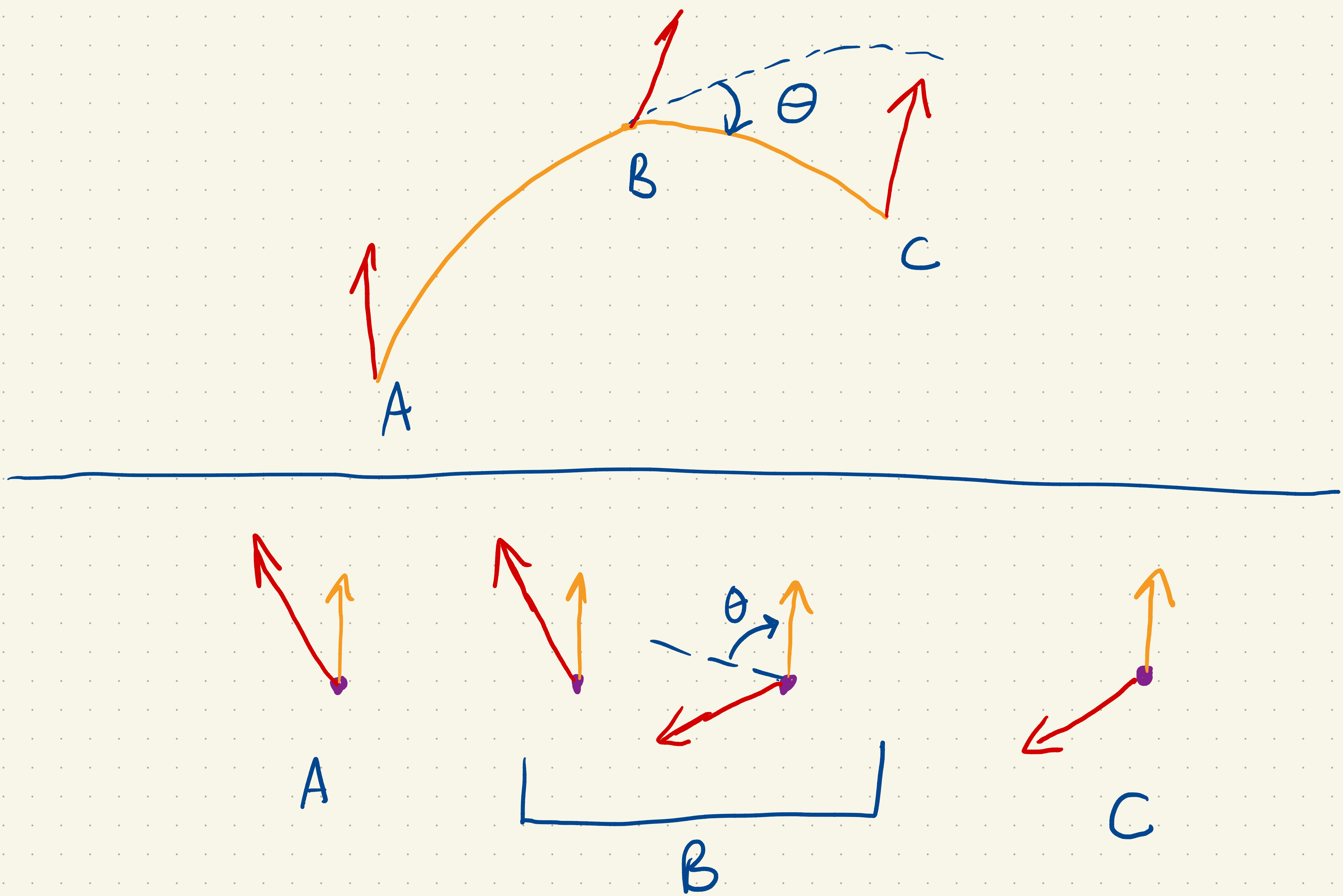
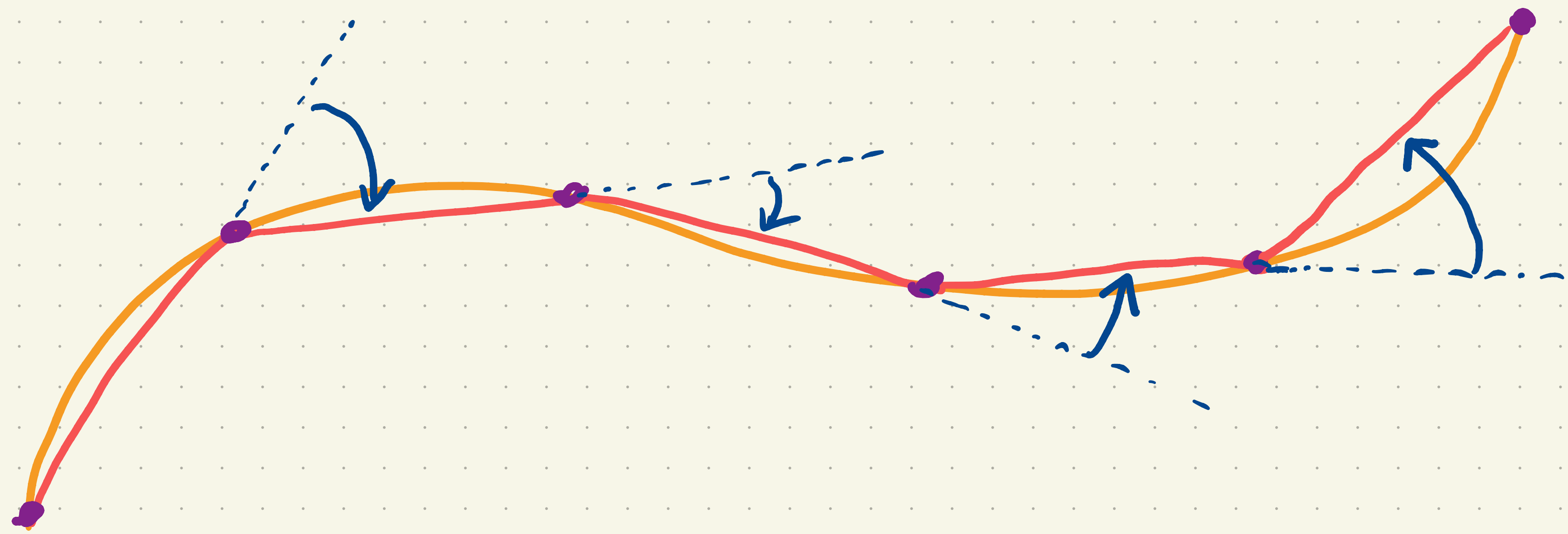




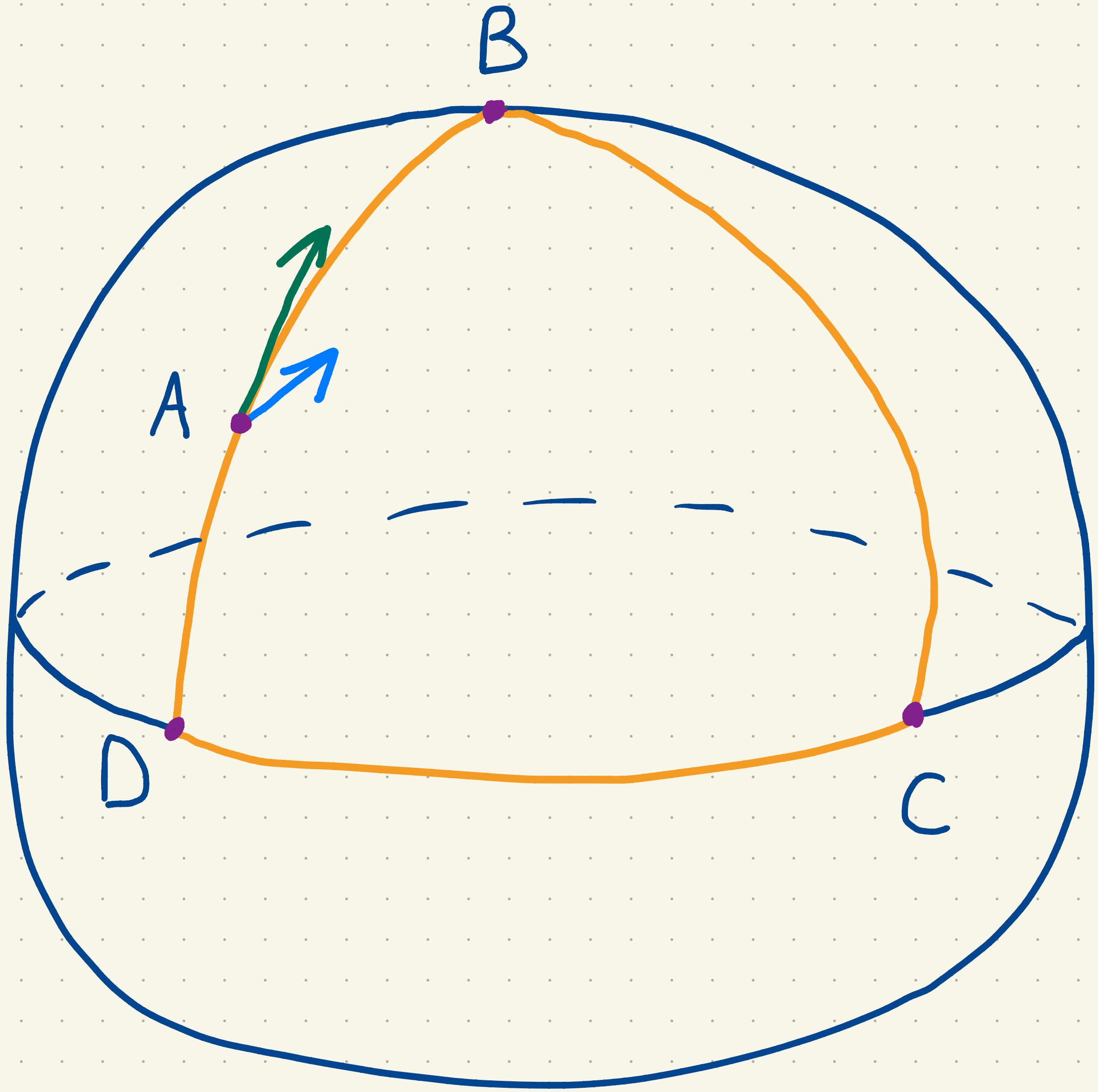
Symmetry Dragging (AKA Parallel Transport)



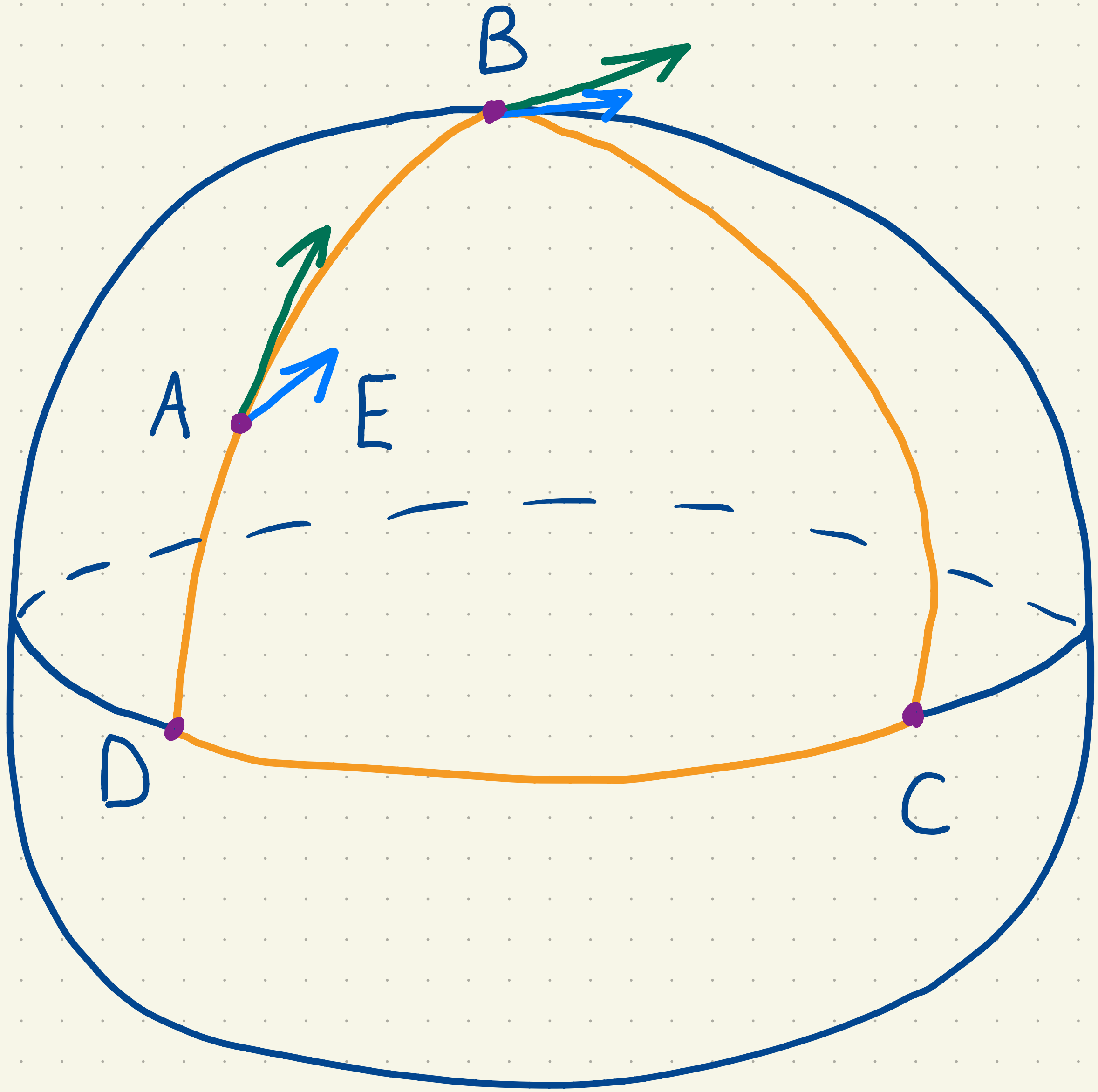
Symmetry Dragging

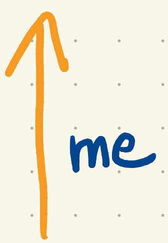


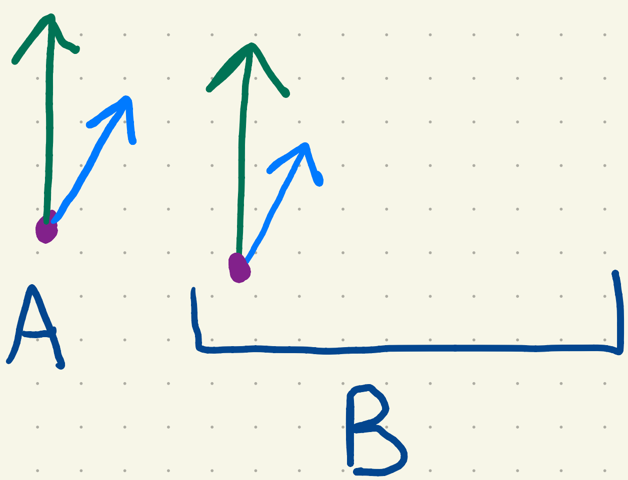
Rotation Anomaly



Rotation Anomaly







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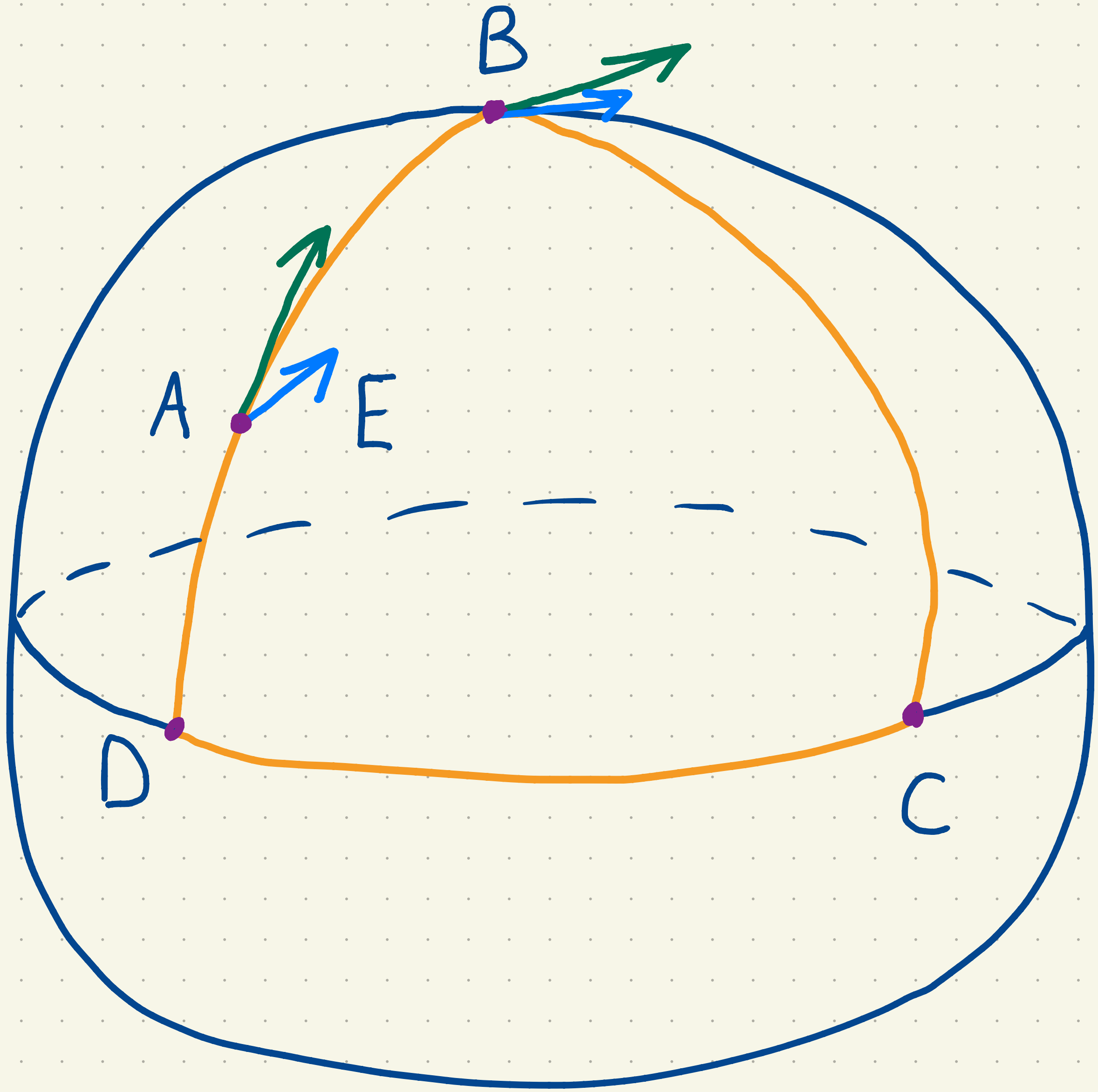

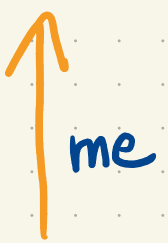

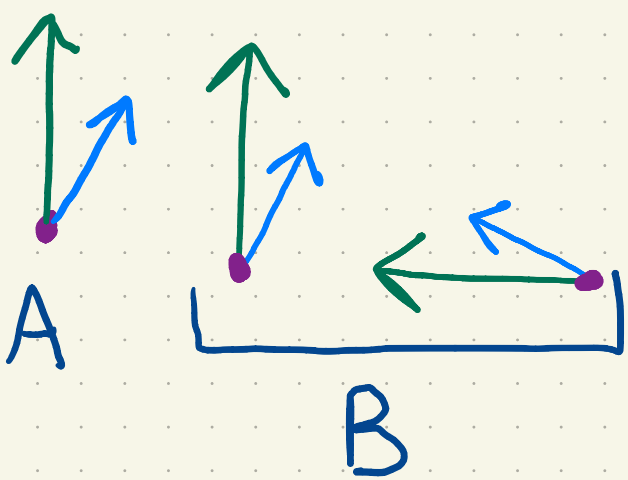

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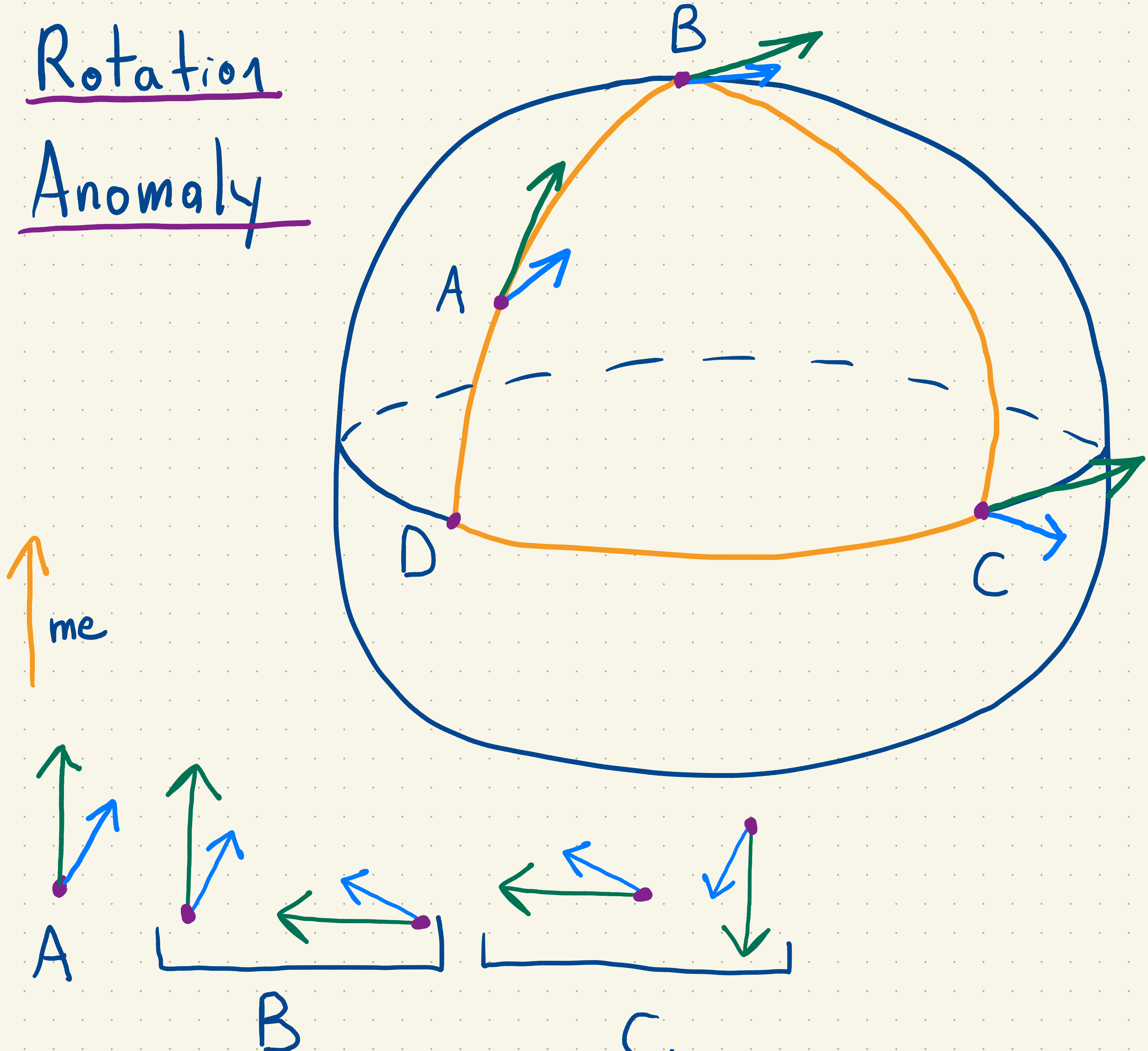
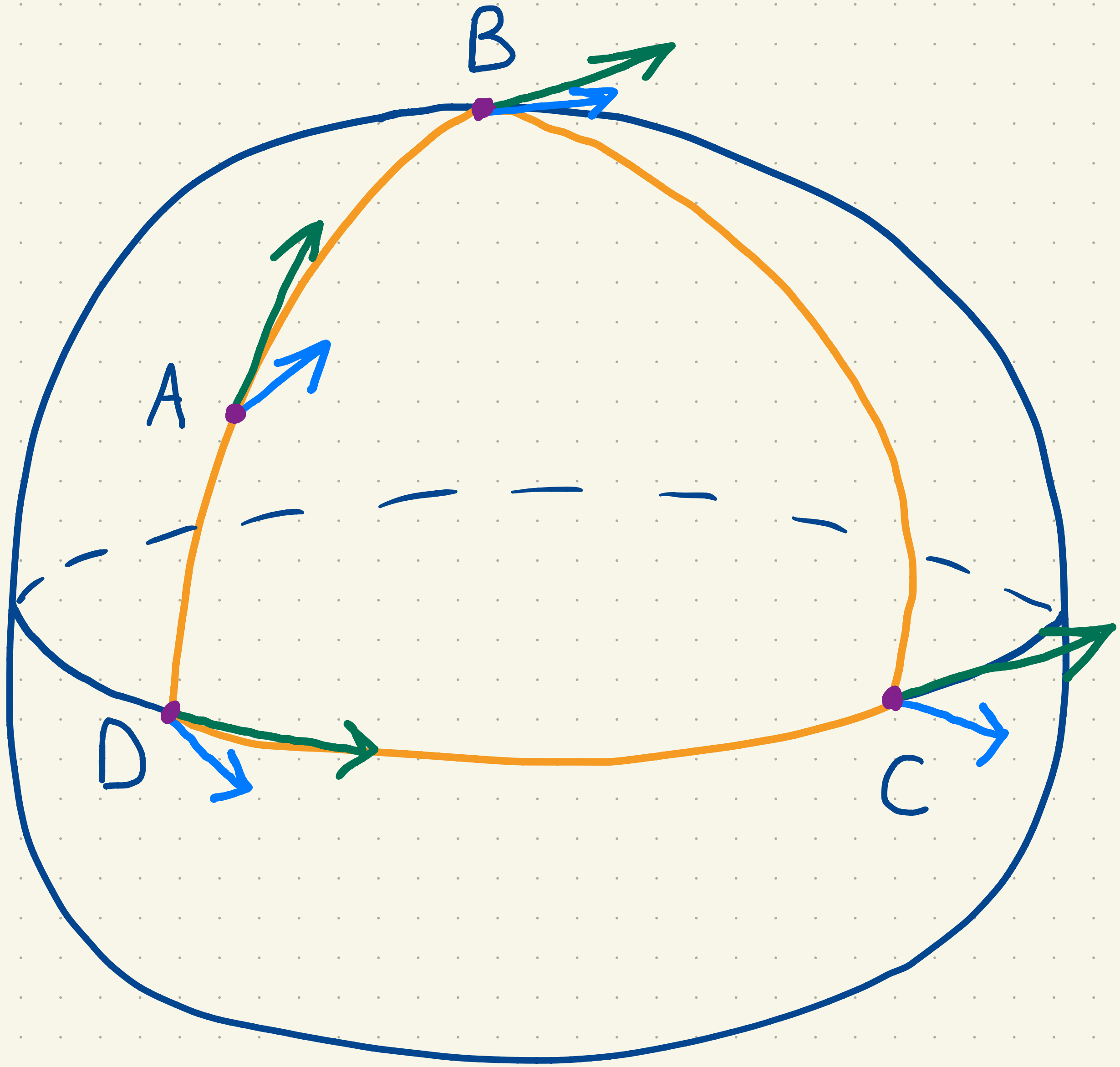
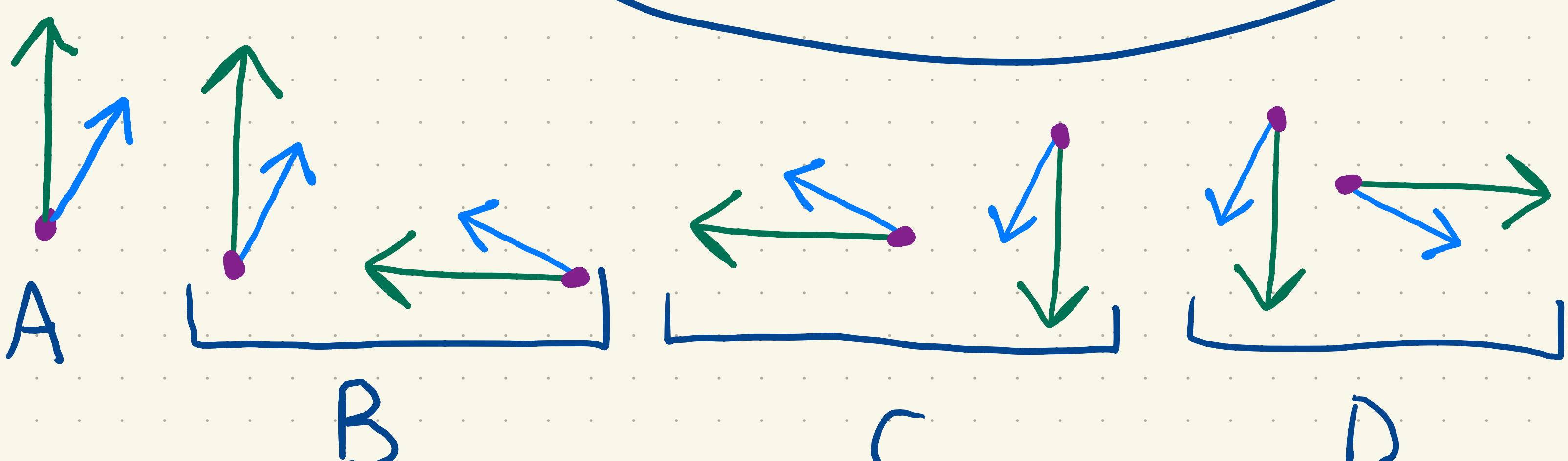


Rotation Anomaly





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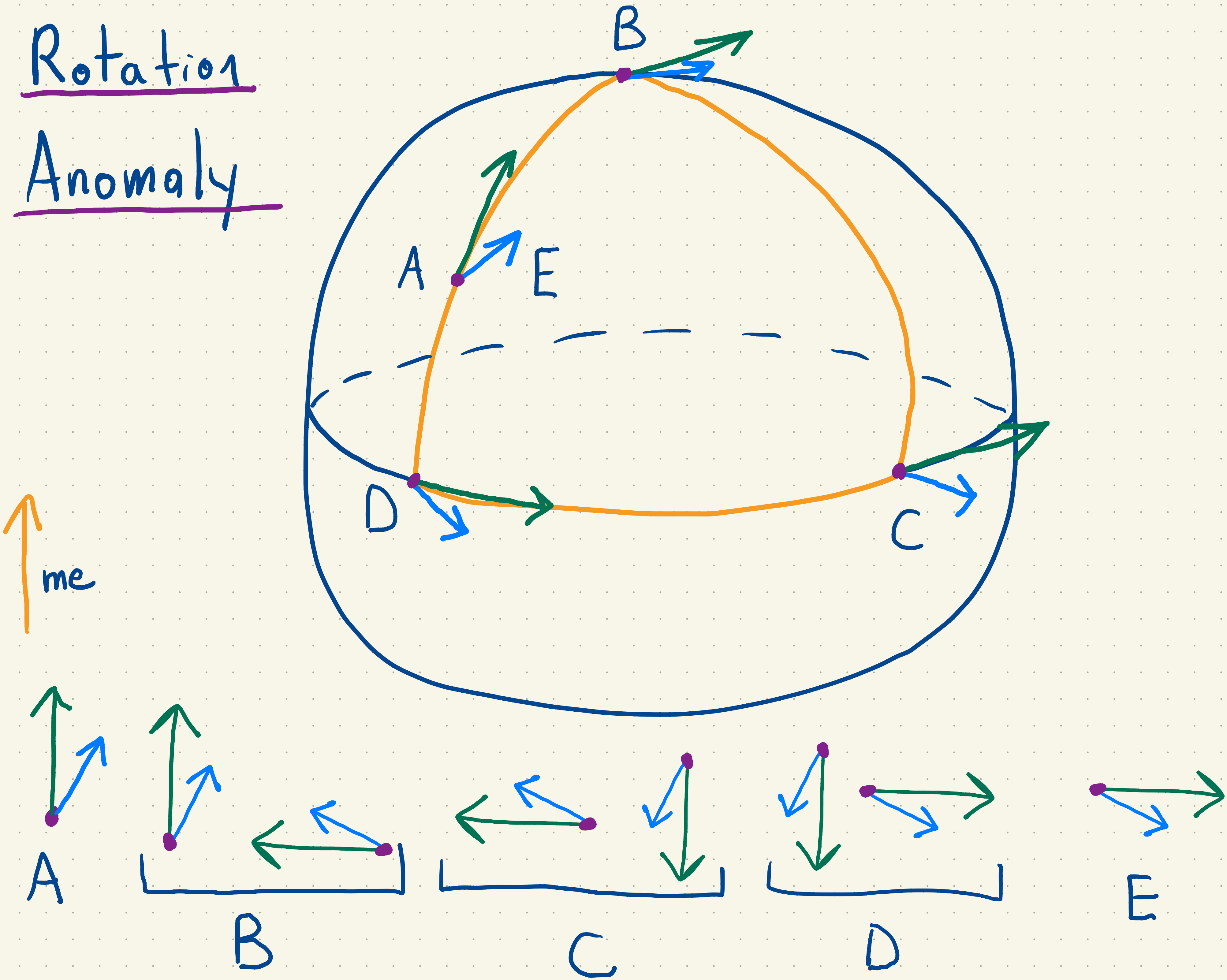
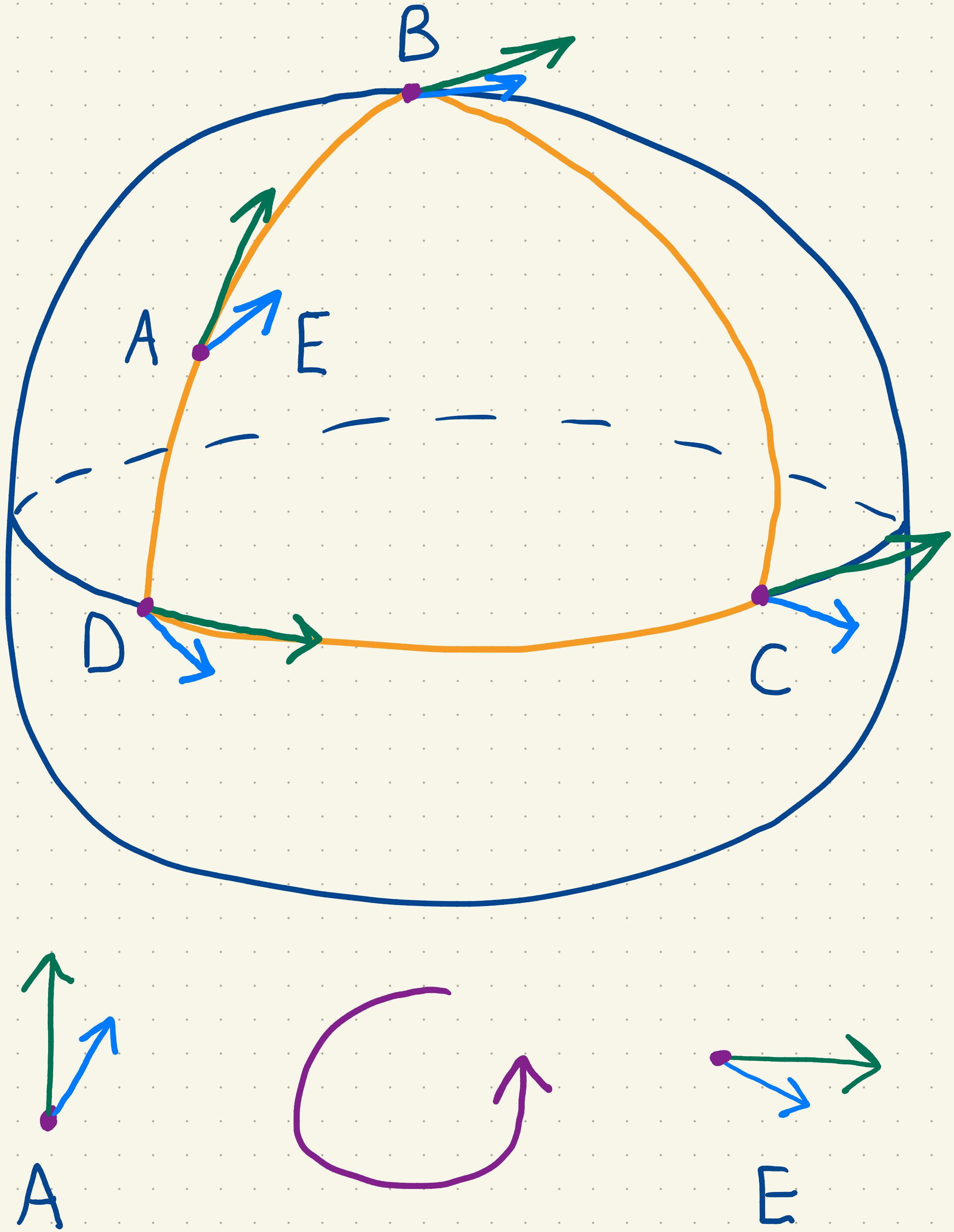
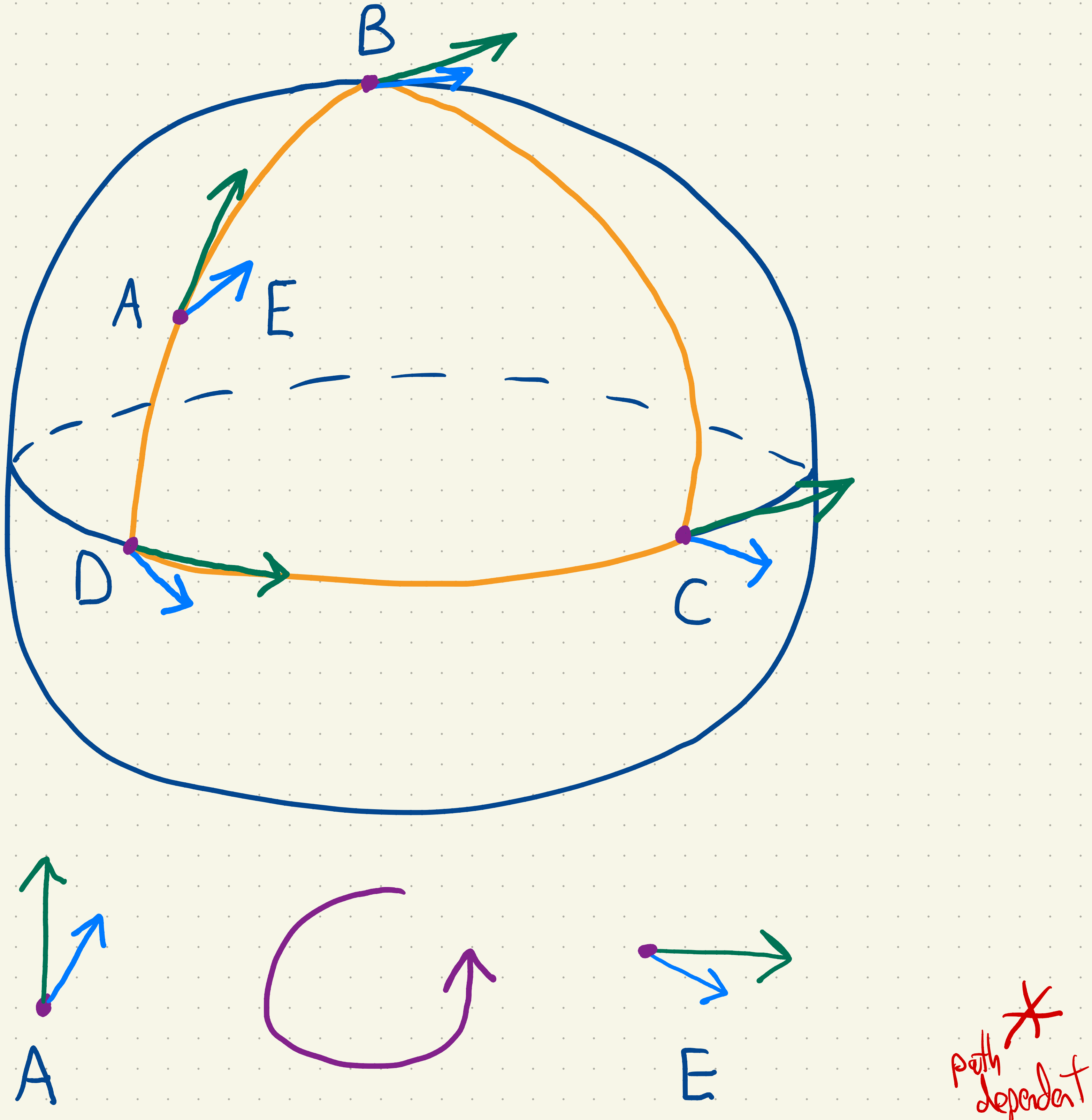


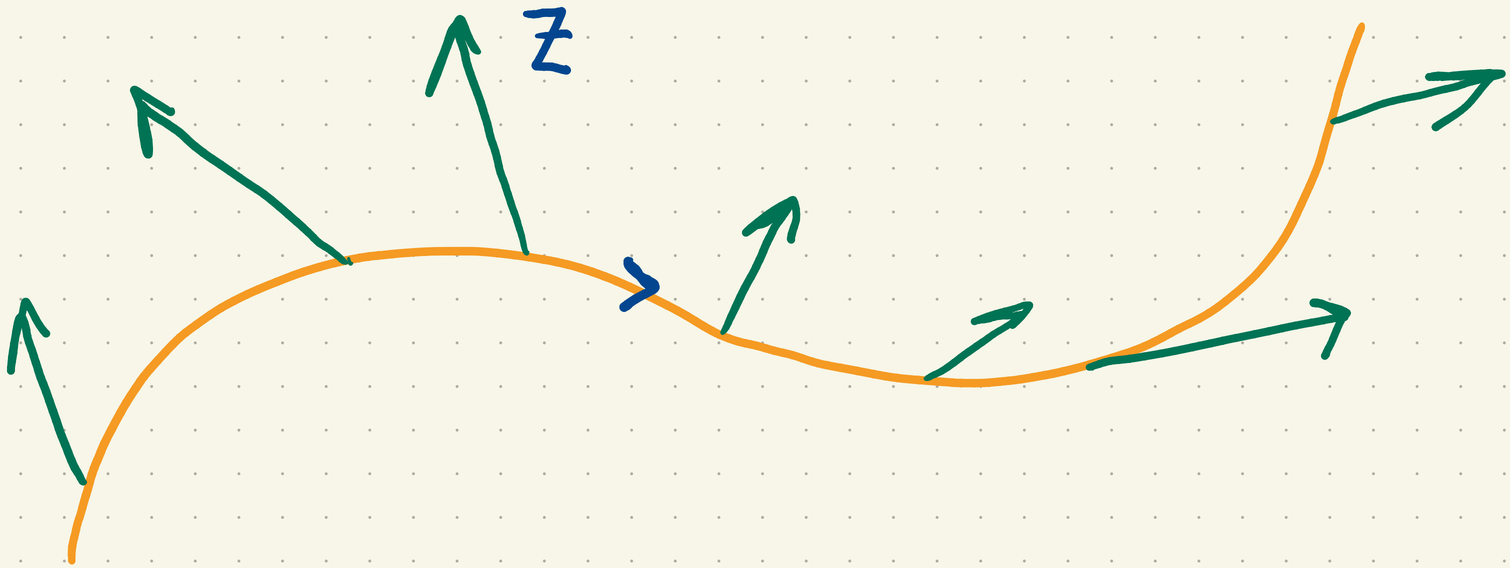
Rotation Anomaly



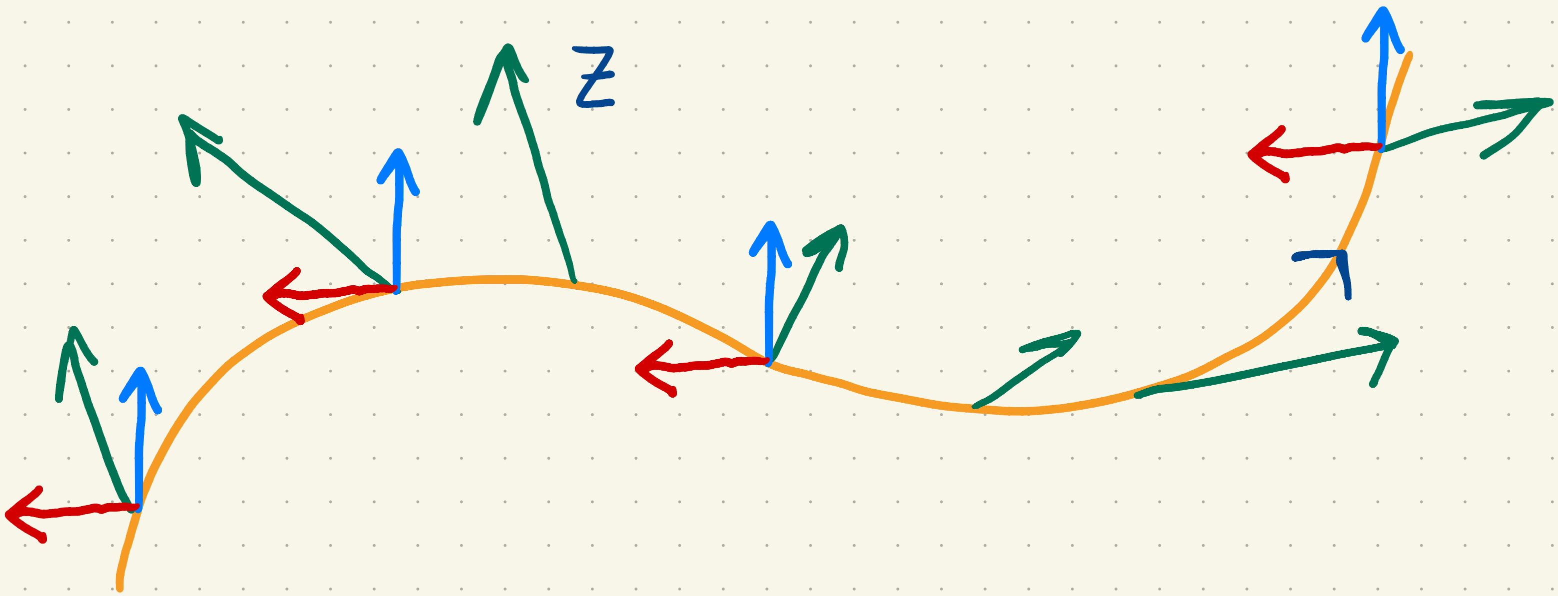
Rotation Anomaly



Symmetry Dragging Lets You Measure Change



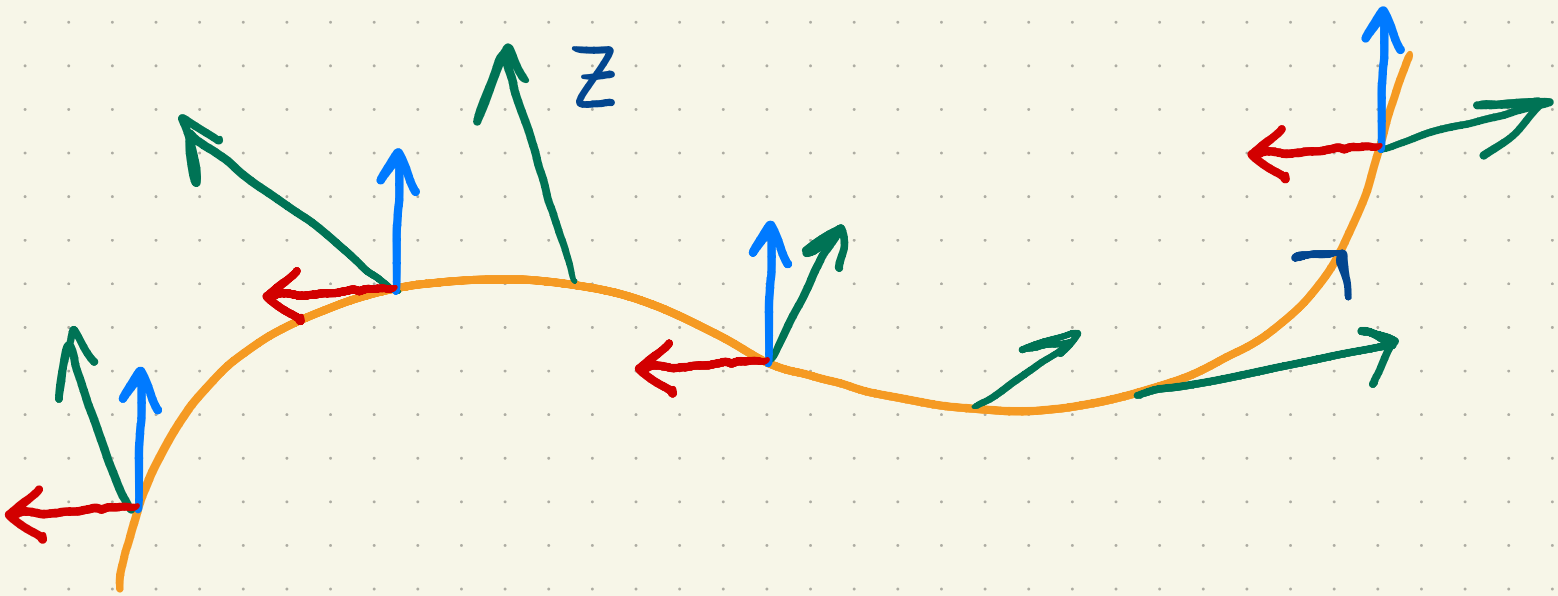
Symmetry Dragging Lets You Measure Change



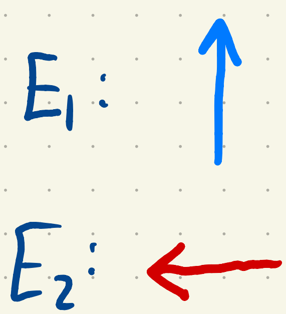
$E_1 : \uparrow$

$E_2 : \leftarrow$

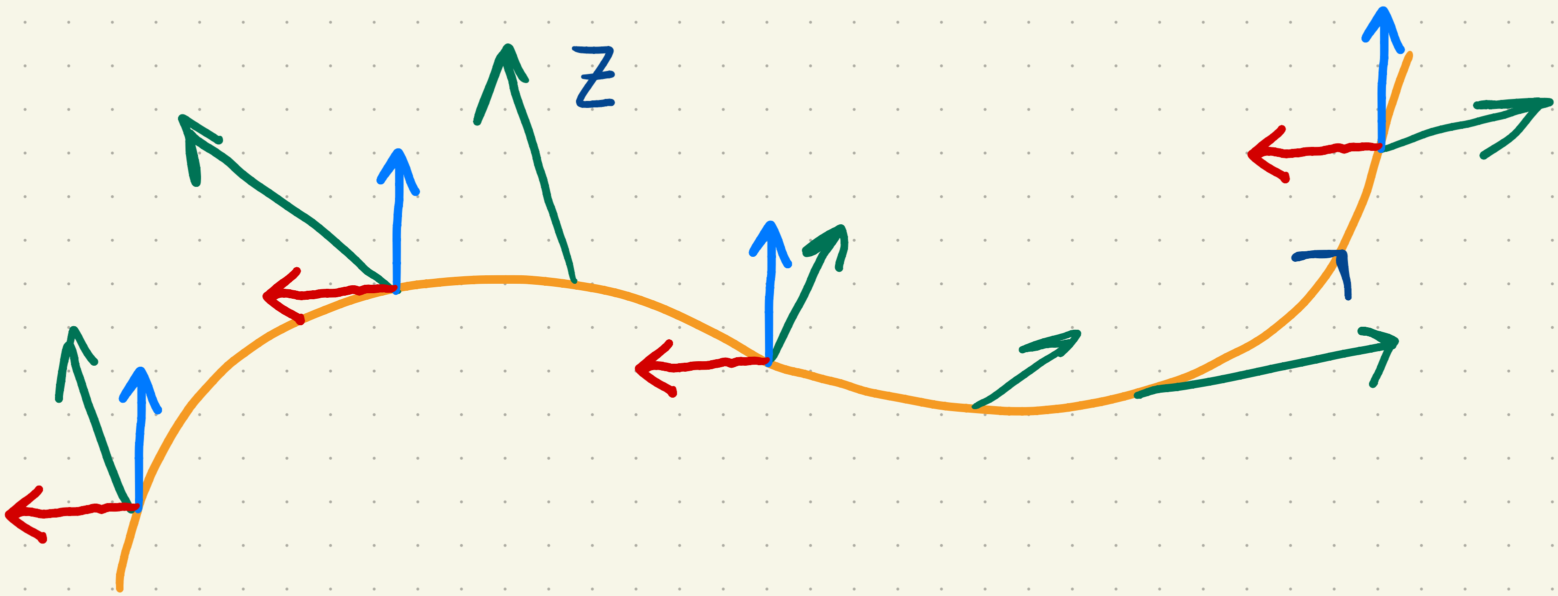
Symmetry Dragging Lets You Measure Change



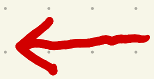
$$Z = Z^1 E_1 + Z^2 E_2$$



Symmetry Dragging Lets You Measure Change



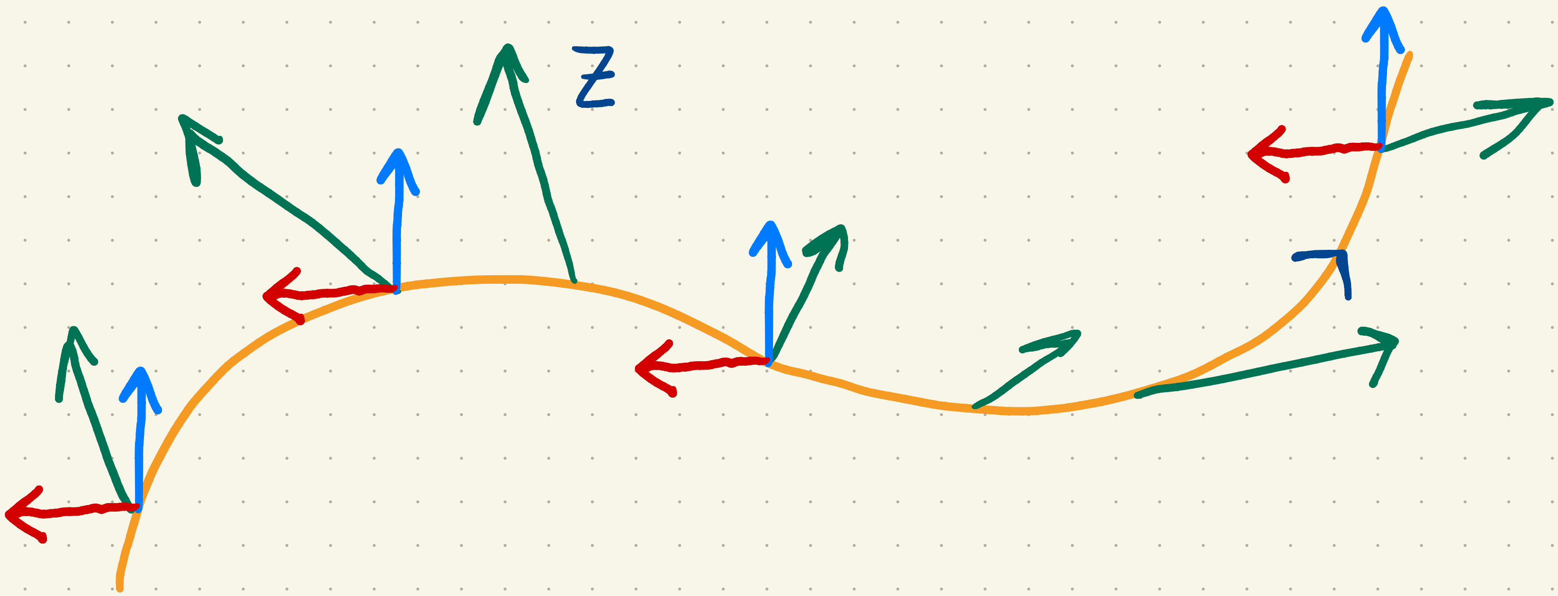
E_1: 

E_2: 

$$Z = Z^1 E_1 + Z^2 E_2$$

$$\nabla_{\dot{\gamma}} Z = \dot{Z}^1 E_1 + \dot{Z}^2 E_2$$

Symmetry Dragging Lets You Measure Change



E_1 :

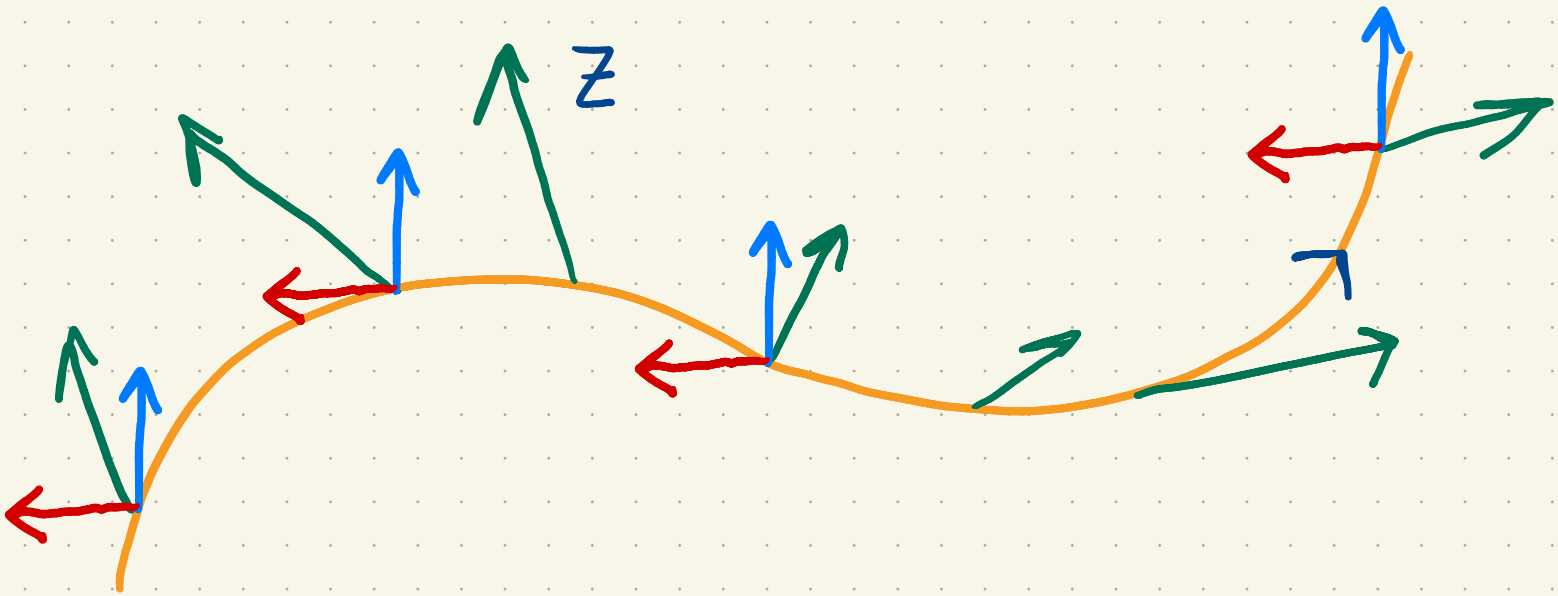
E_2 :

$$Z = Z^1 E_1 + Z^2 E_2$$

$$\nabla_{\dot{\gamma}} Z = \dot{Z}^1 E_1 + \dot{Z}^2 E_2$$

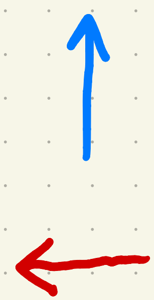
$$\nabla_X Z = \nabla_{\dot{\gamma}} Z$$

Symmetry Dragging Lets You Measure Change



E_1:

E_2:



$$Z = Z^1 E_1 + Z^2 E_2$$

$$\nabla_{\dot{\gamma}} Z = \dot{Z}^1 E_1 + \dot{Z}^2 E_2$$

$$\nabla_X Z = \nabla_{\dot{\gamma}} Z$$

cumbersome