An Unhelpful Introduction to Electricity & Magnetism

Part I: Really Unhelpful

Nov 10, 2020

David Maxwell

UAF Mathematics

Background

- Gauss' Law:

$$\vec{\nabla} \cdot \vec{E} = \rho$$

Background

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$$\vec{\nabla} \cdot \vec{E} = \rho$$

- "part of shape of now" = "stuff density"

Background

- Gauss' Law:

$$\vec{\nabla} \cdot \vec{E} = \rho$$

- "part of shape of now" = "stuff density"

- How do you describe the stuff in the universe before it has a shape?

No:
- angles
- lengths
- volumes
- dot products

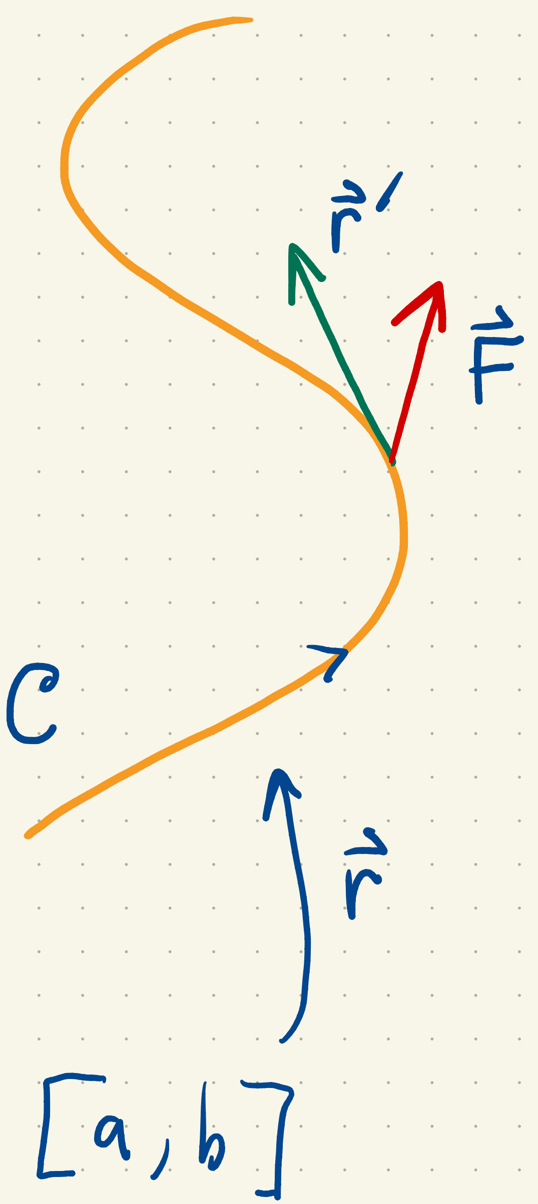
Outline

- Today: exterior derivative
  covectors and their kin

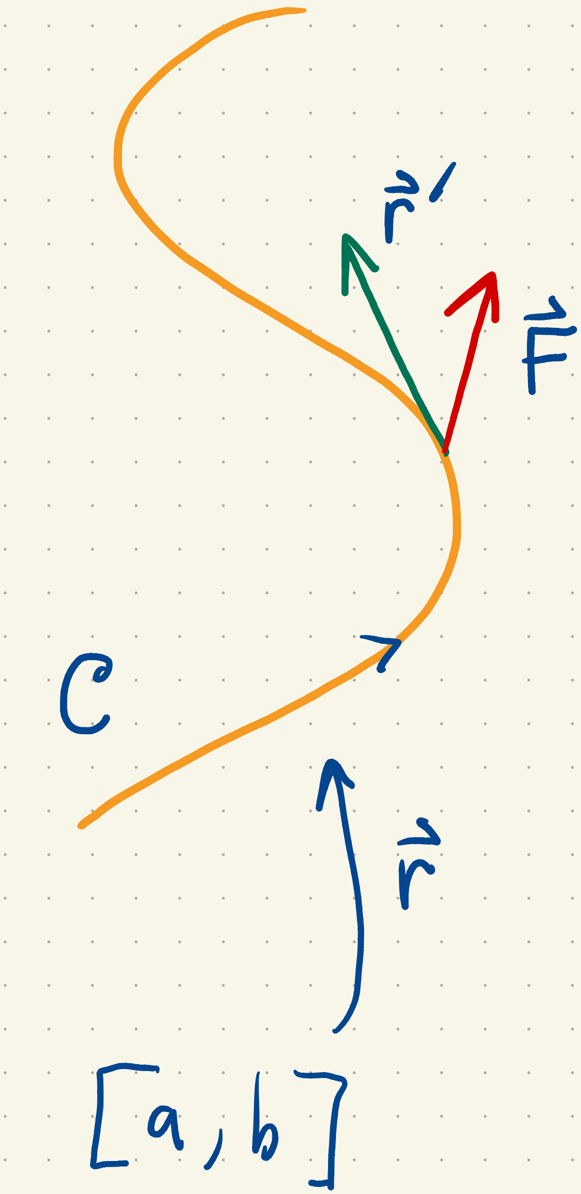
11/17 • What is the object described by electromagnetism?

11/24 • Where do Maxwell's equations come from?

Line Integrals

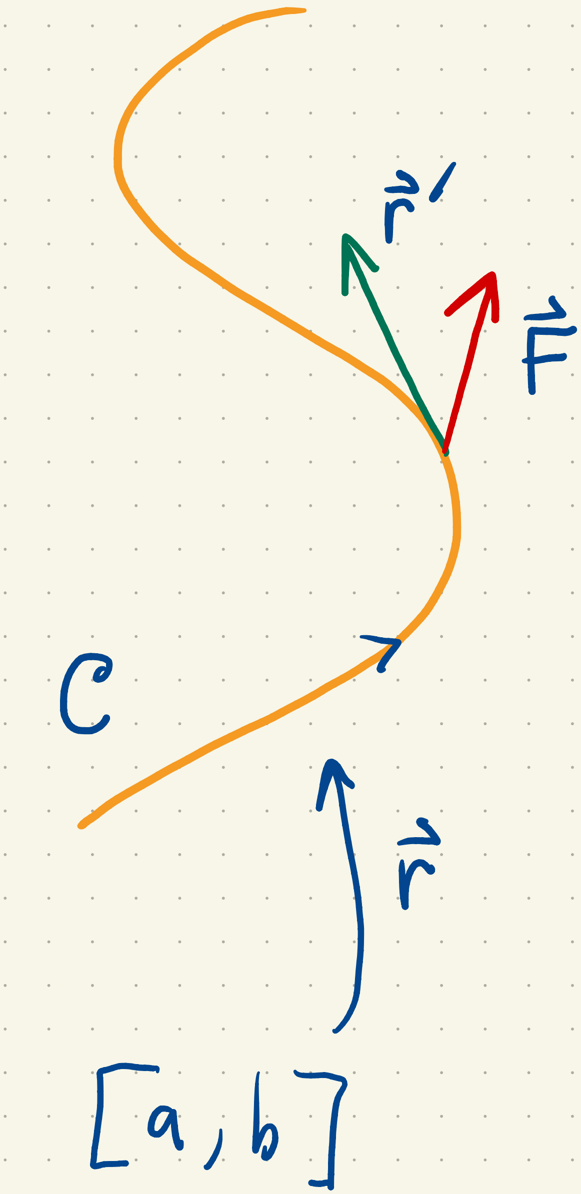


Line Integrals



$$\int_C \vec{F} \cdot d\vec{r}$$

Line Integrals

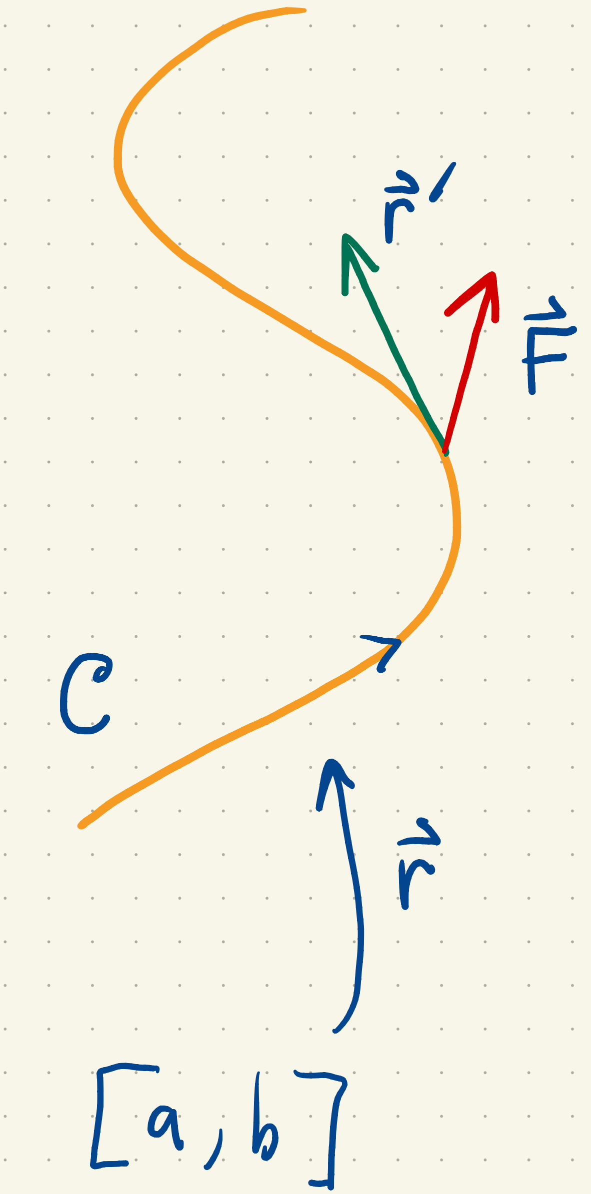


$$\int_{C} \vec{F} \cdot d\vec{r}$$

$$\updownarrow$$

$$\int_{a}^{b} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

Line Integrals



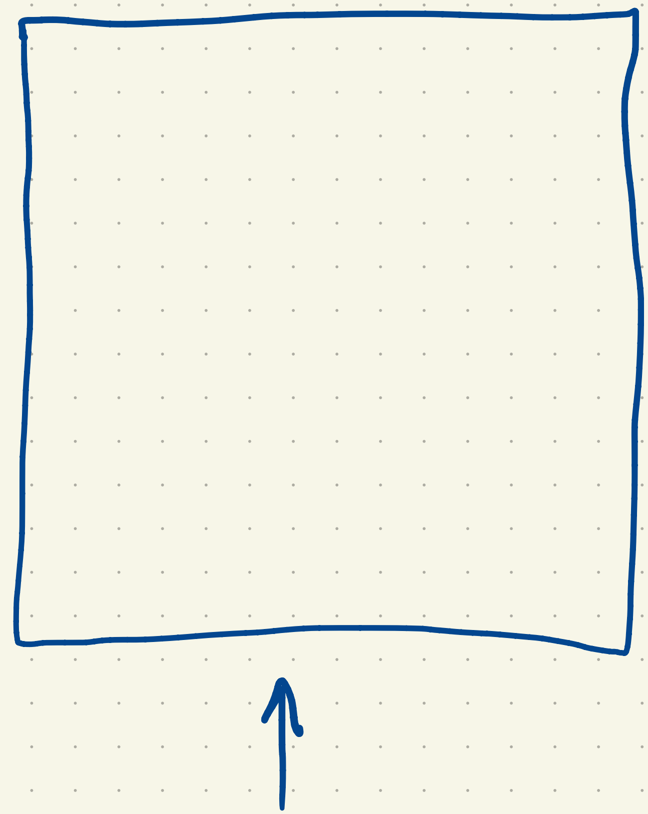
$$\int_C \vec{F} \cdot d\vec{r}$$

$$\Updownarrow$$

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

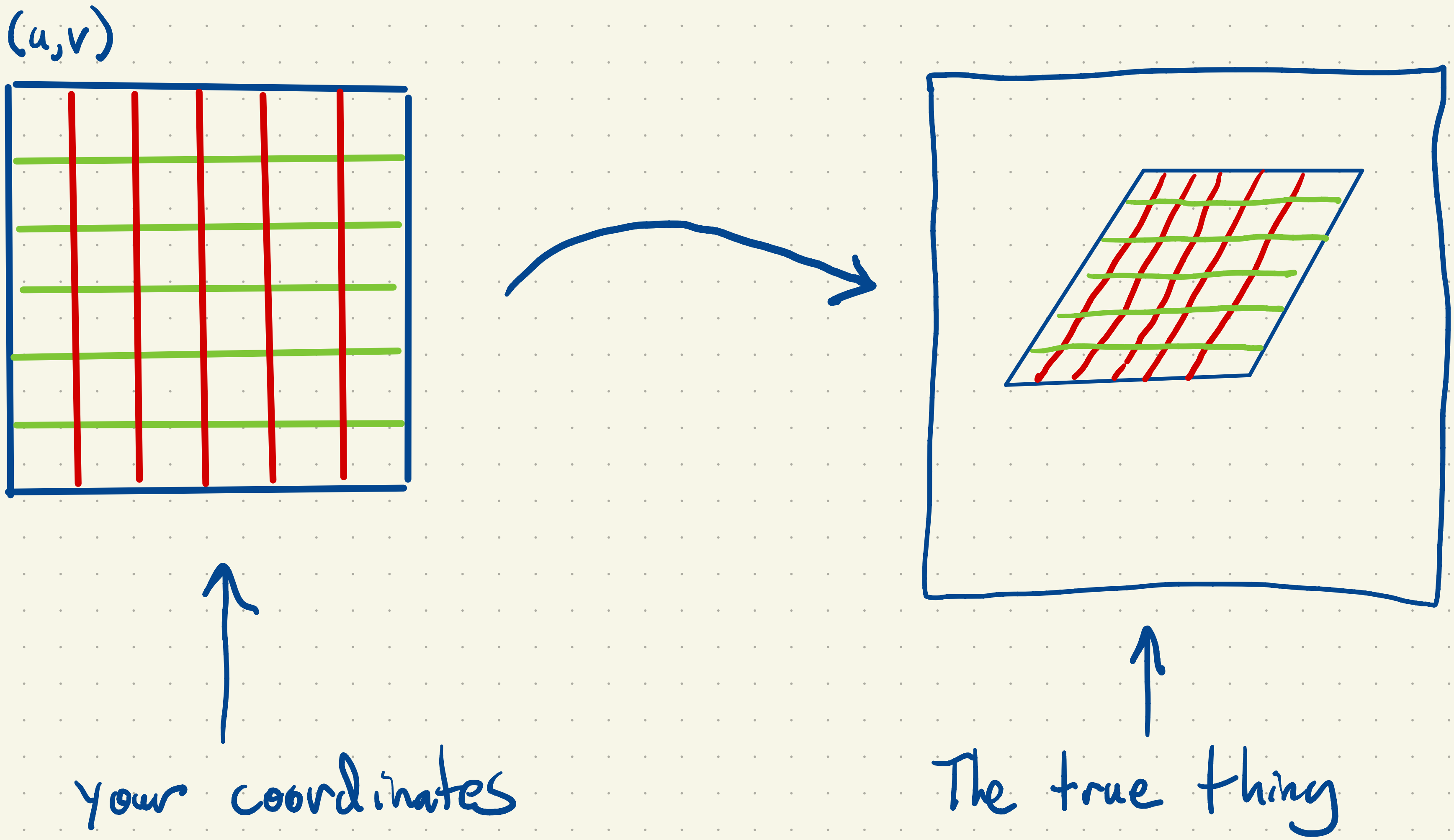
uh oh

Coordinates

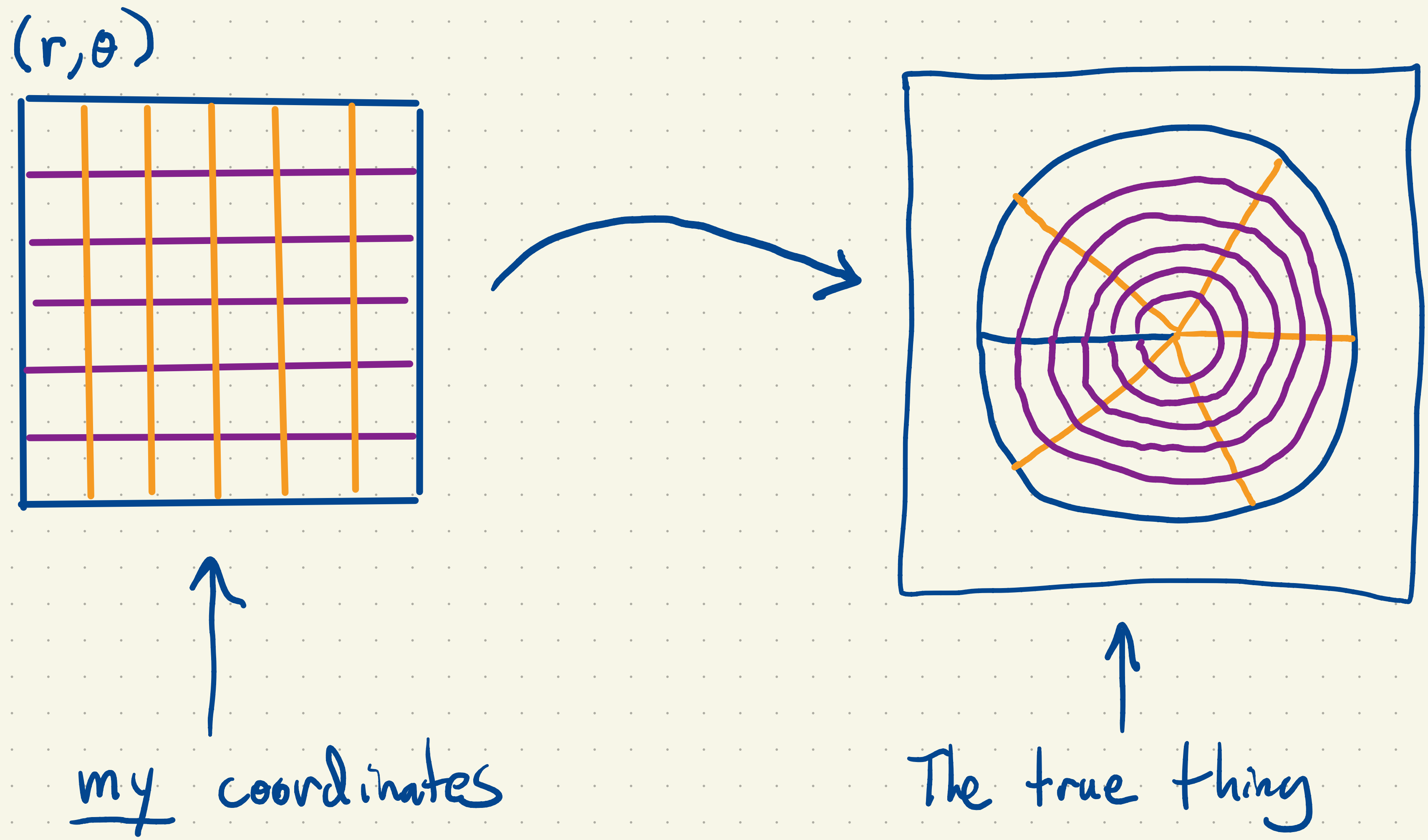


The true thing

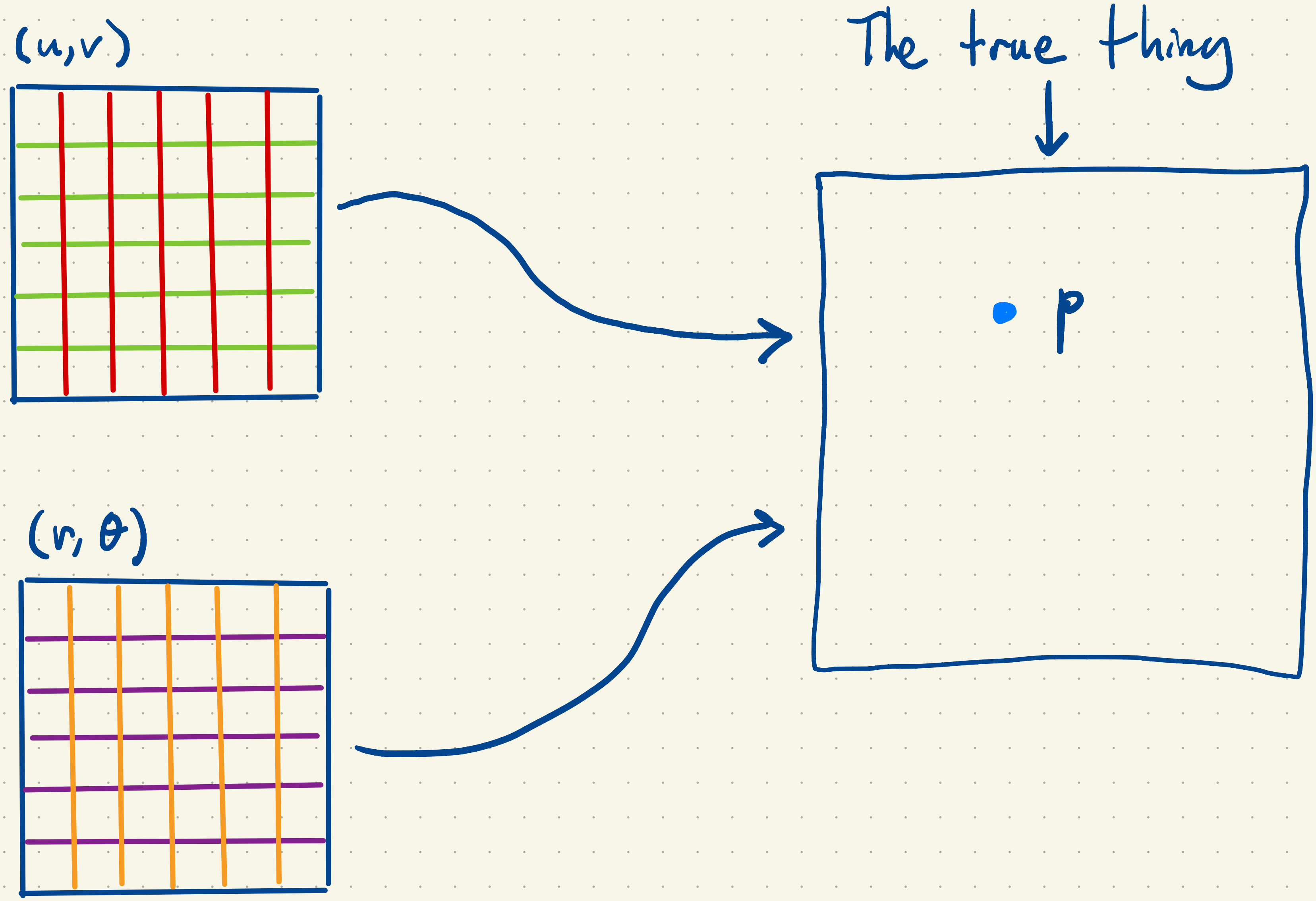
Coordinates



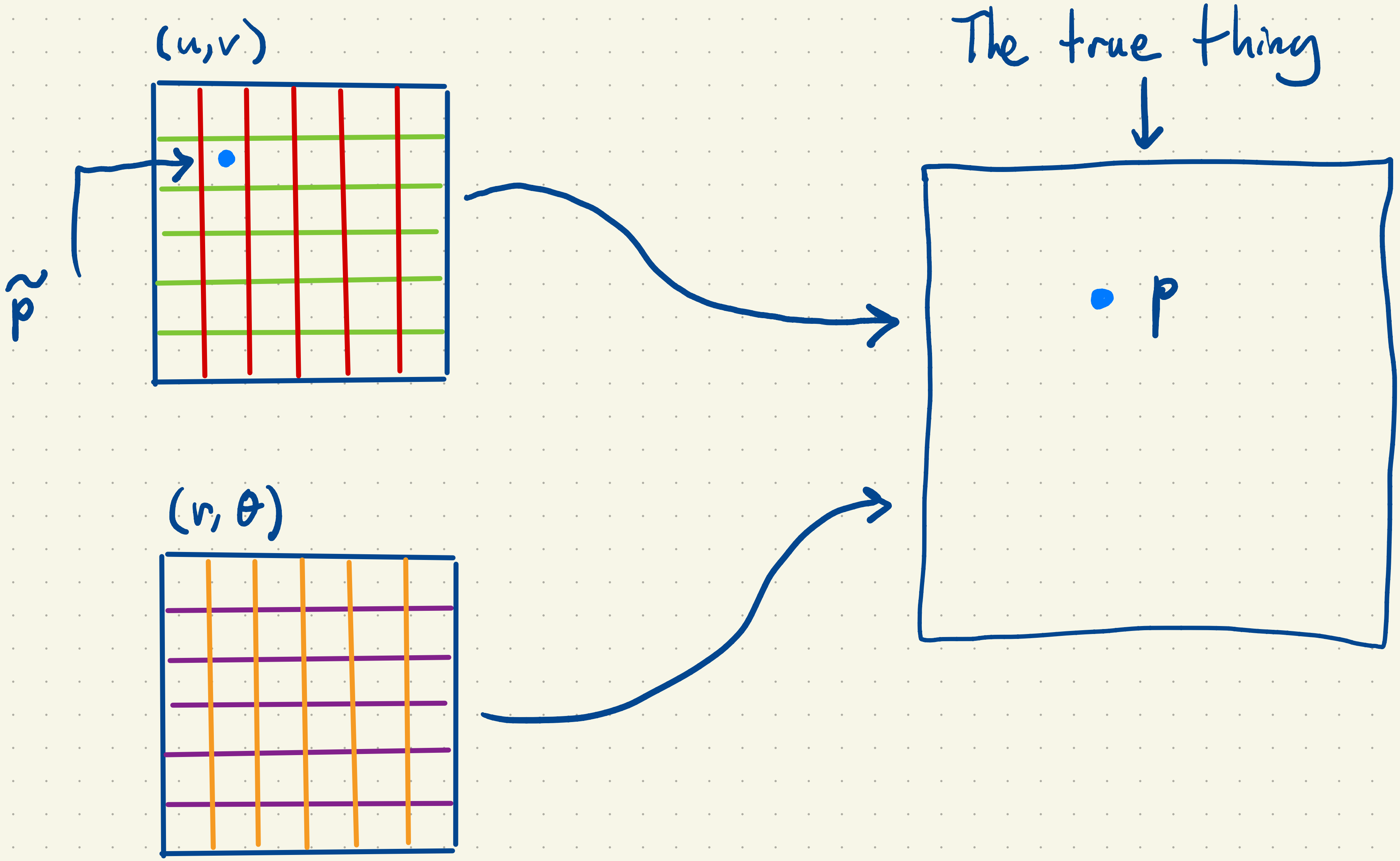
Coordinates



Change of Coordinates



Change of Coordinates



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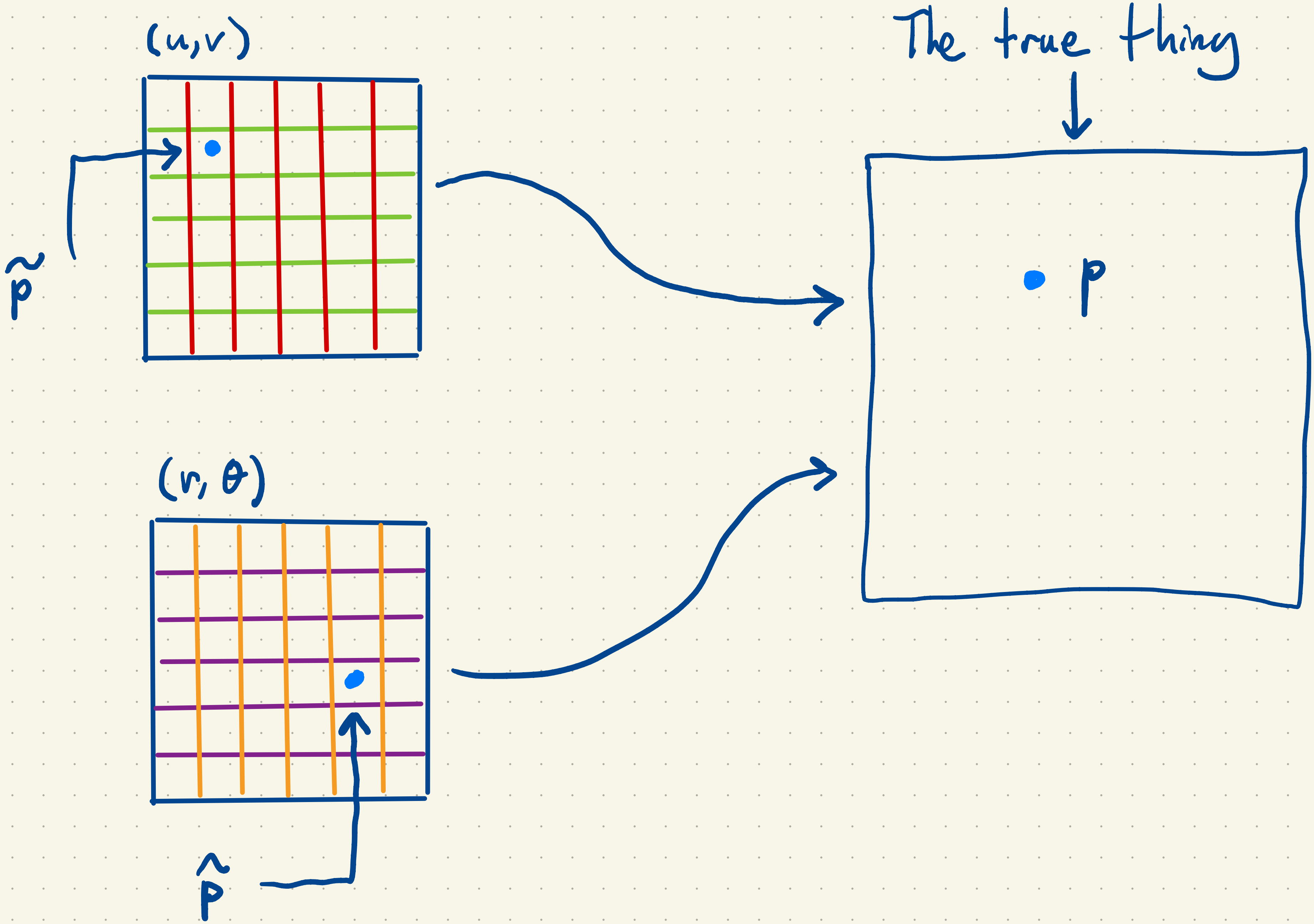

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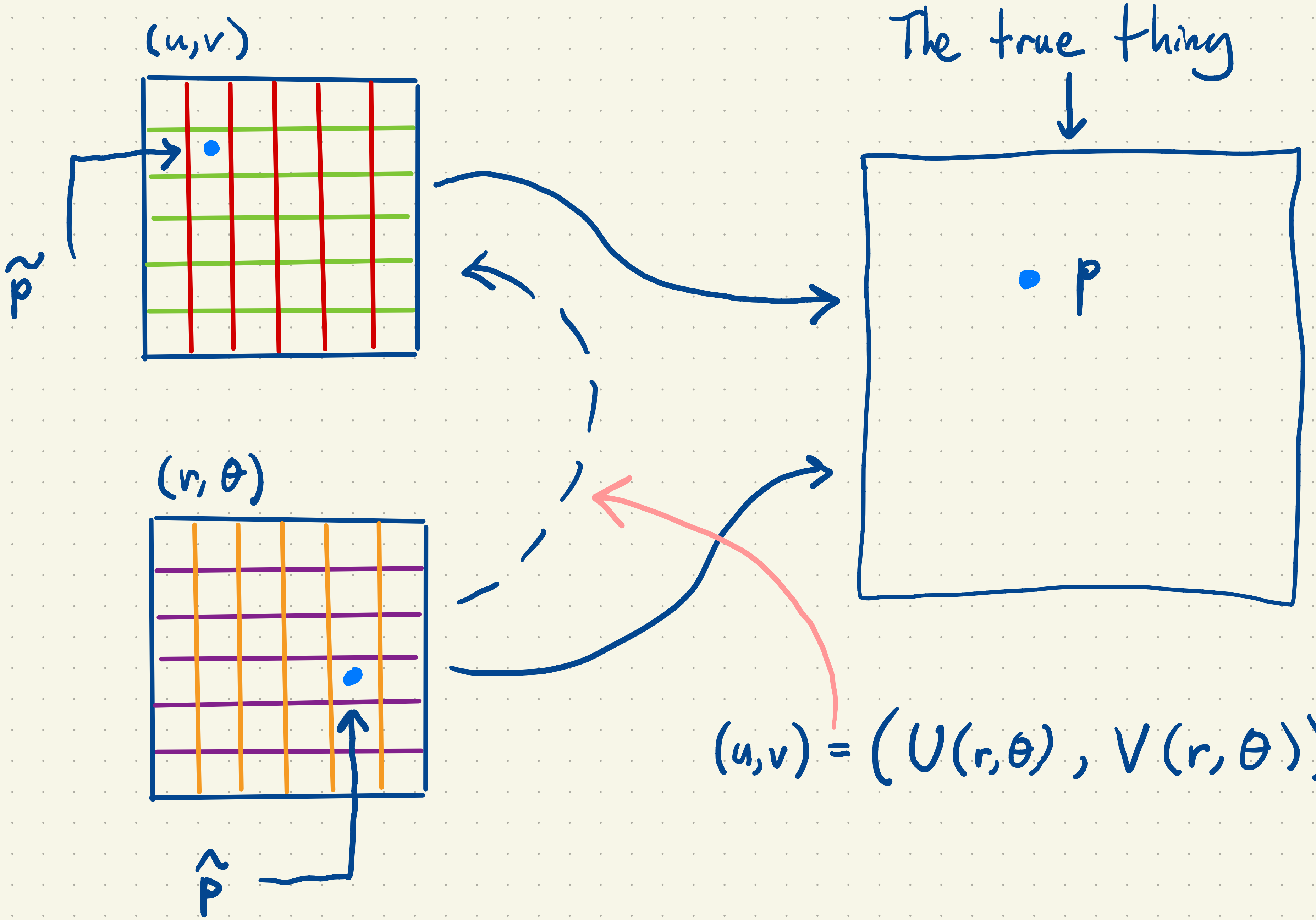
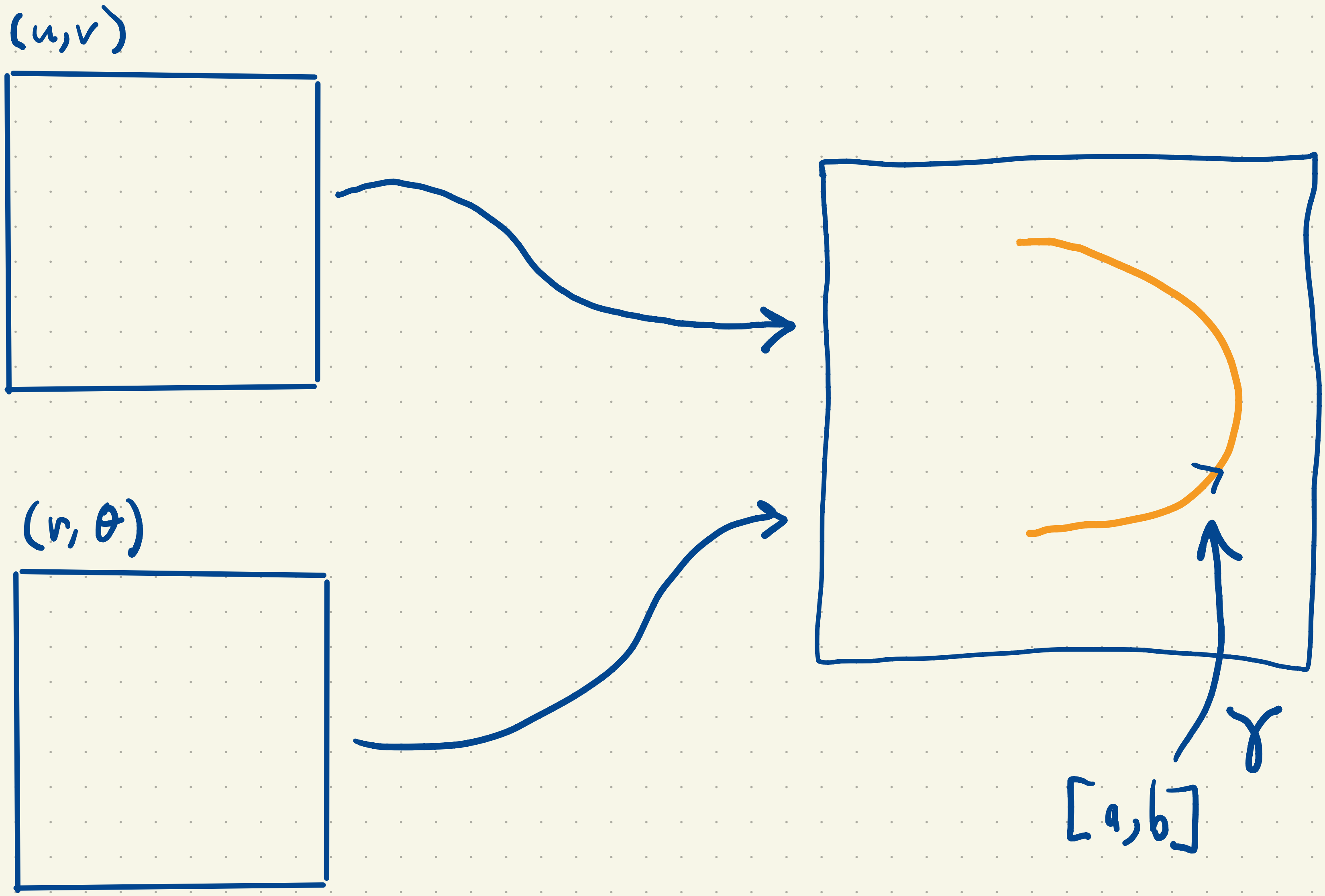
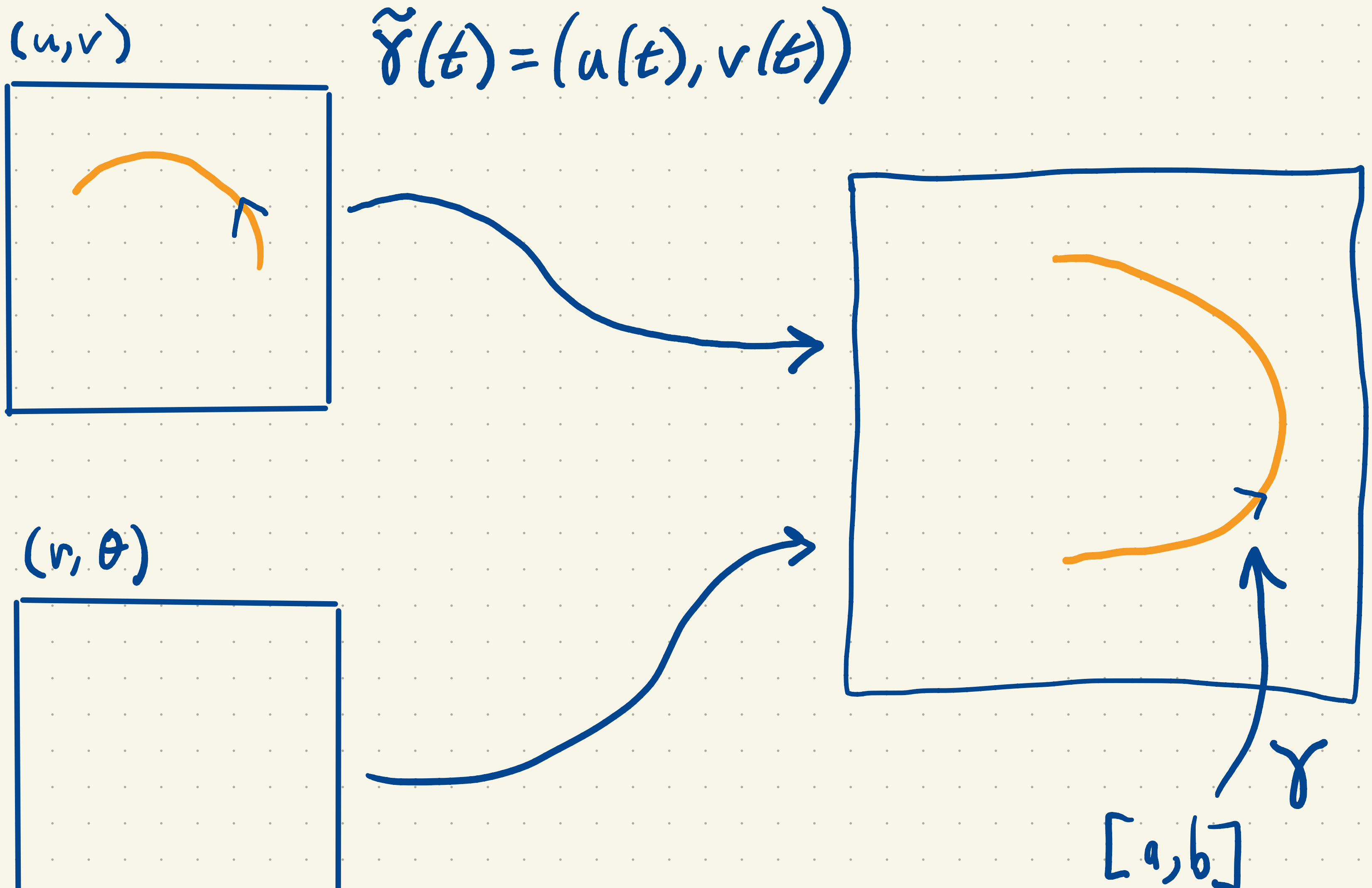


$$(u,v) = \big(U(r,\theta),\ V(r,\theta)\big)$$

Tangent Vector = Infinitesimal Curve

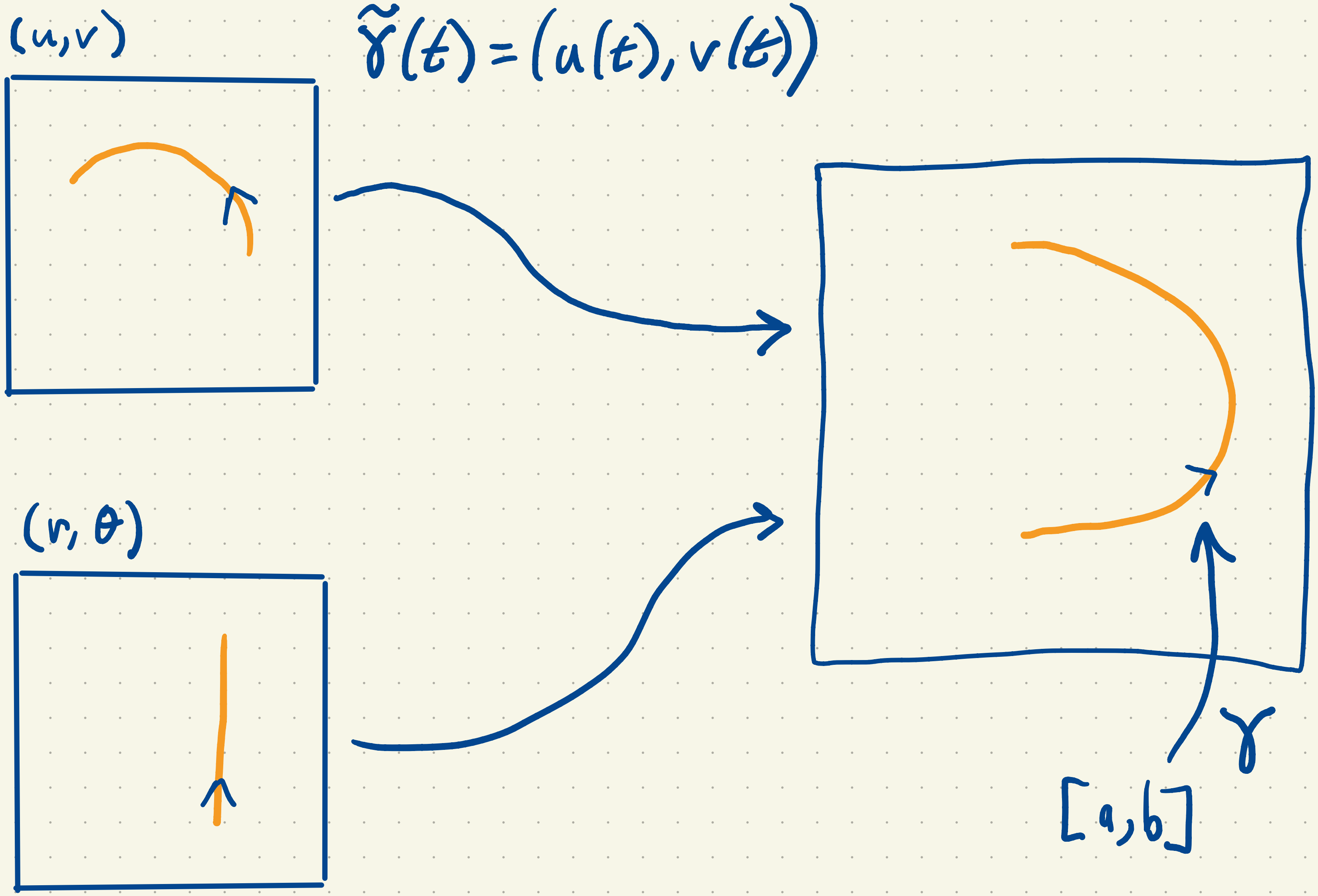


Tangent Vector = Infinitesimal Curve



Tangent Vector = Infinitesimal Curve

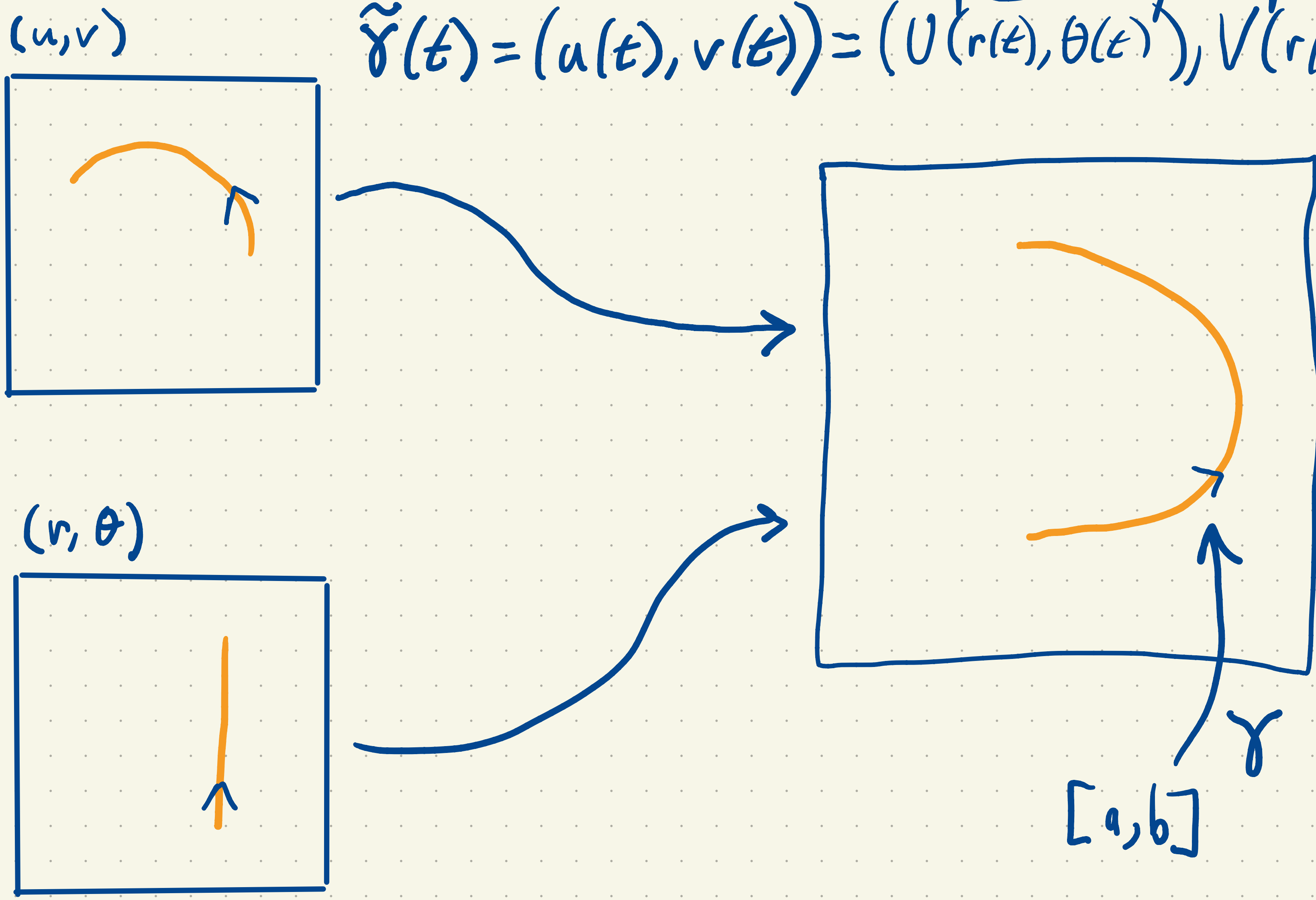
$$\tilde{\gamma}(t) = (u(t), v(t))$$



$$\hat{\gamma}(t) = (r(t), \theta(t))$$

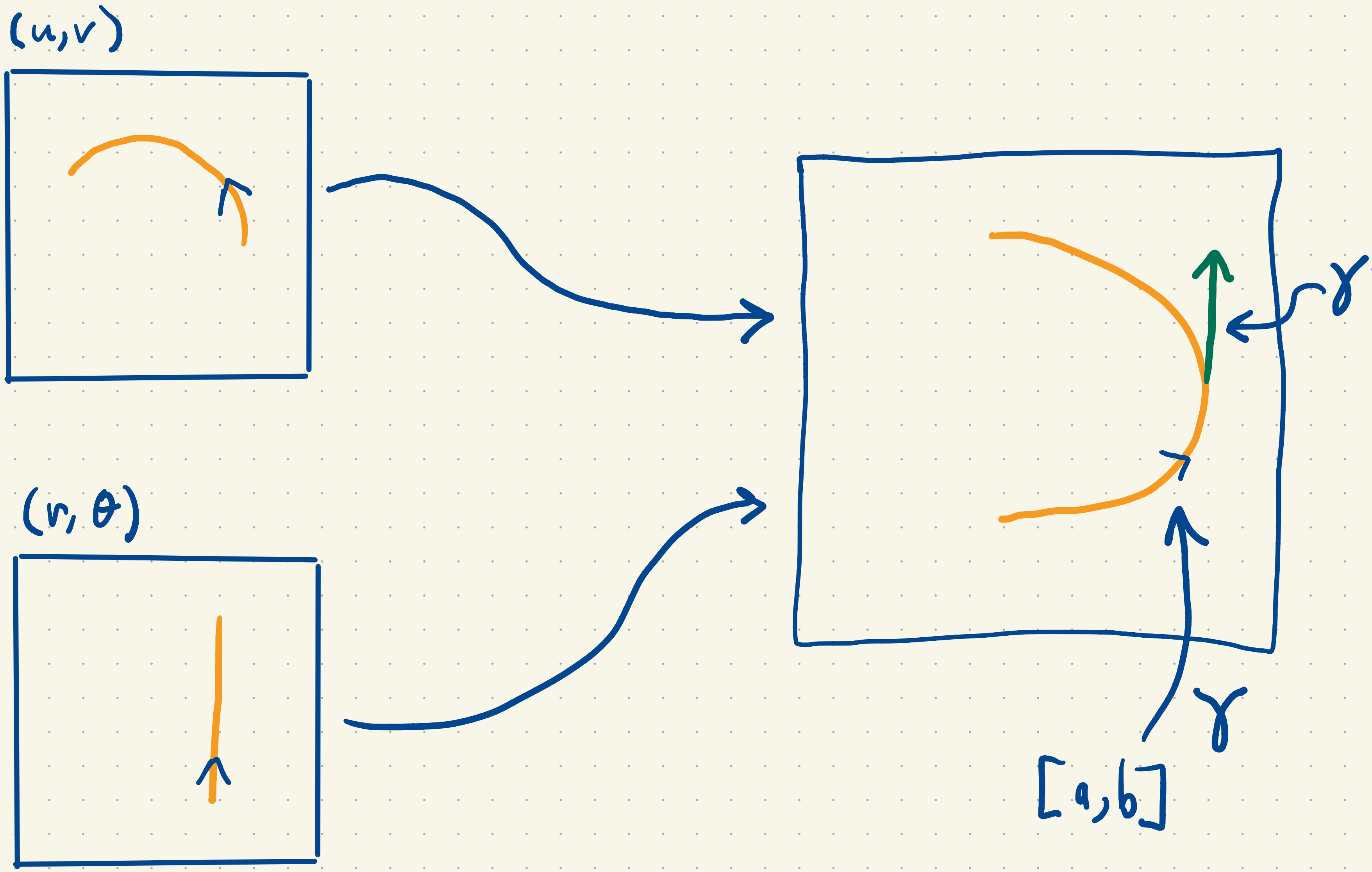
Tangent Vector = Infinitesimal Curve

$$\tilde{\gamma}(t) = (u(t), v(t)) = \left(U(\overbrace{r(t), \theta(t)}^{\hat{\gamma}(t)}), V(\overbrace{r(t), \theta(t)}^{\hat{\gamma}(t)})\right)$$



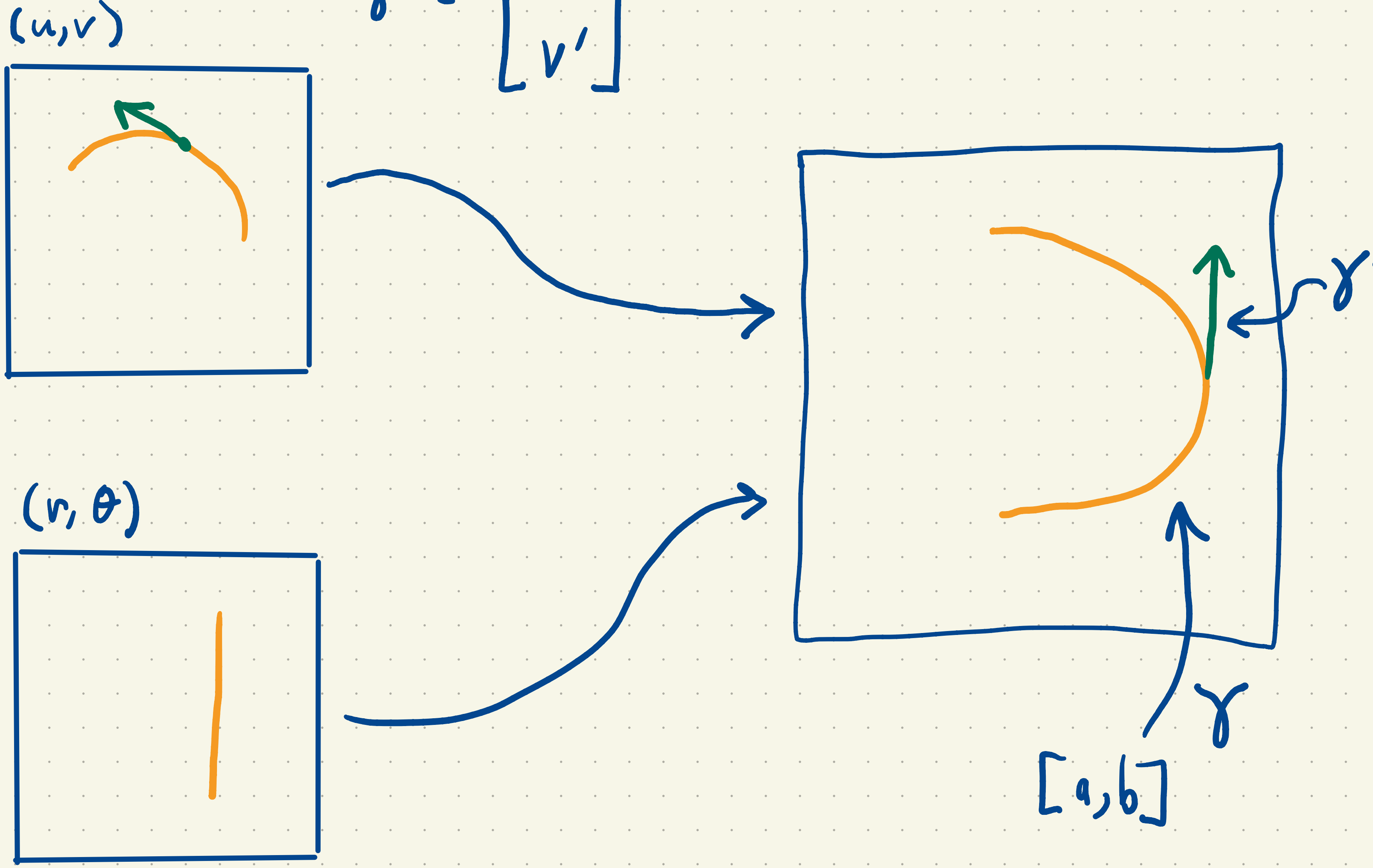
$$\hat{\gamma}(t) = (r(t), \theta(t))$$

Tangent Vector = Infinitesimal Curve

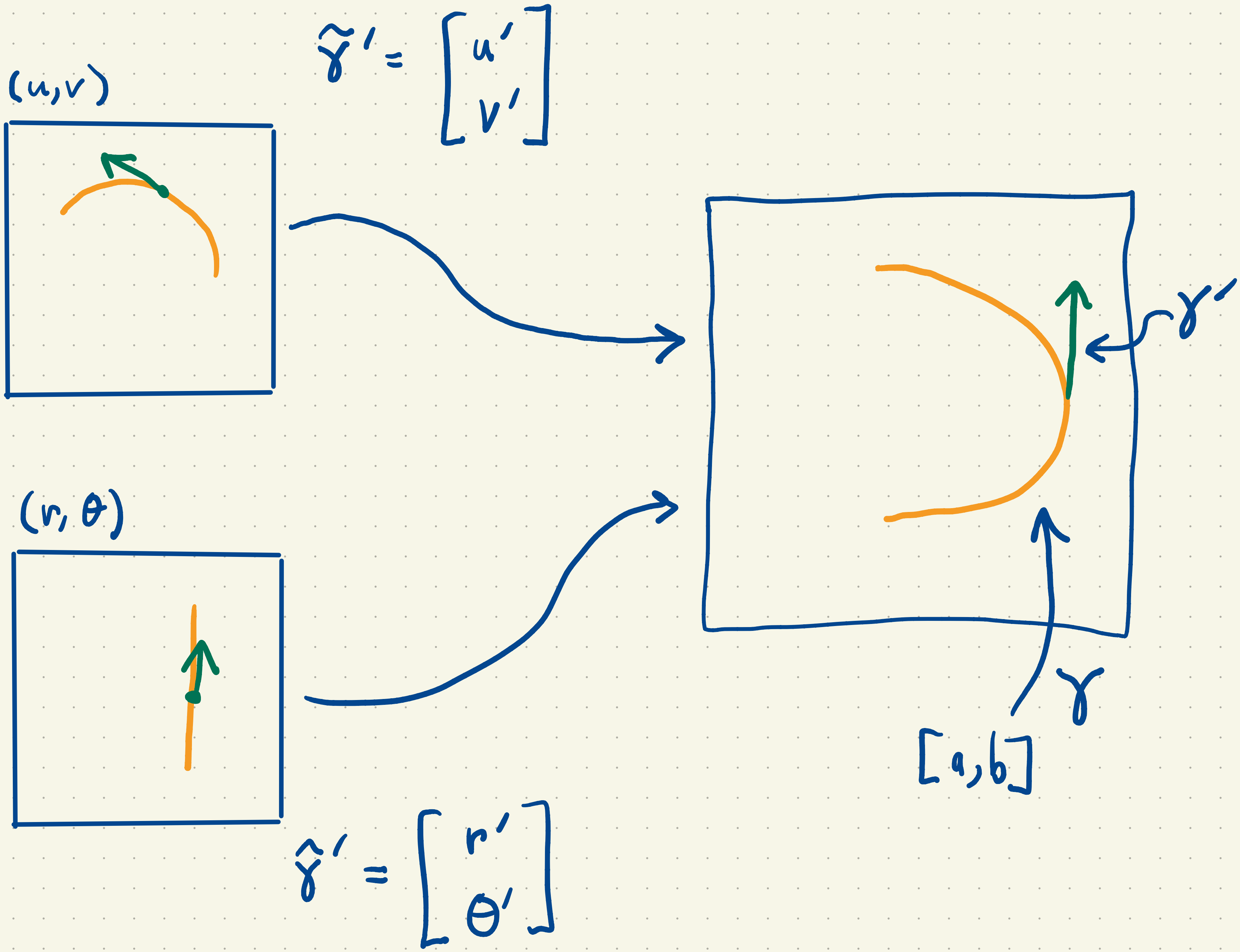


Tangent Vector = Infinitesimal Curve

$$\tilde{\gamma}' = \begin{bmatrix} u' \\ v' \end{bmatrix}$$

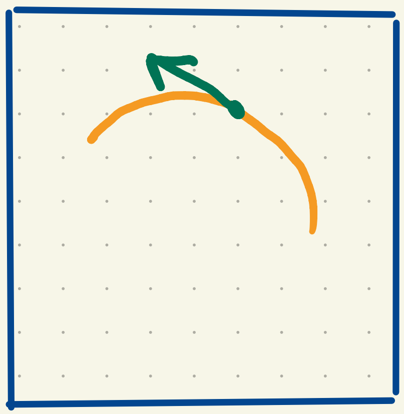


Tangent Vector = Infinitesimal Curve

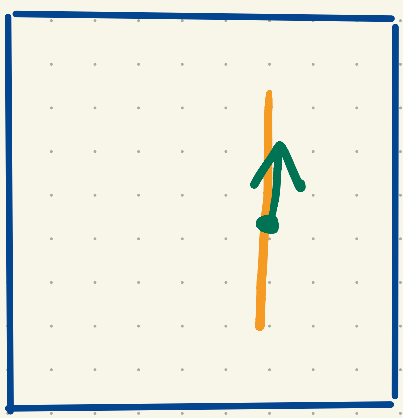


Tangent Vector = Infinitesimal Curve

(u, v)



(r, θ)



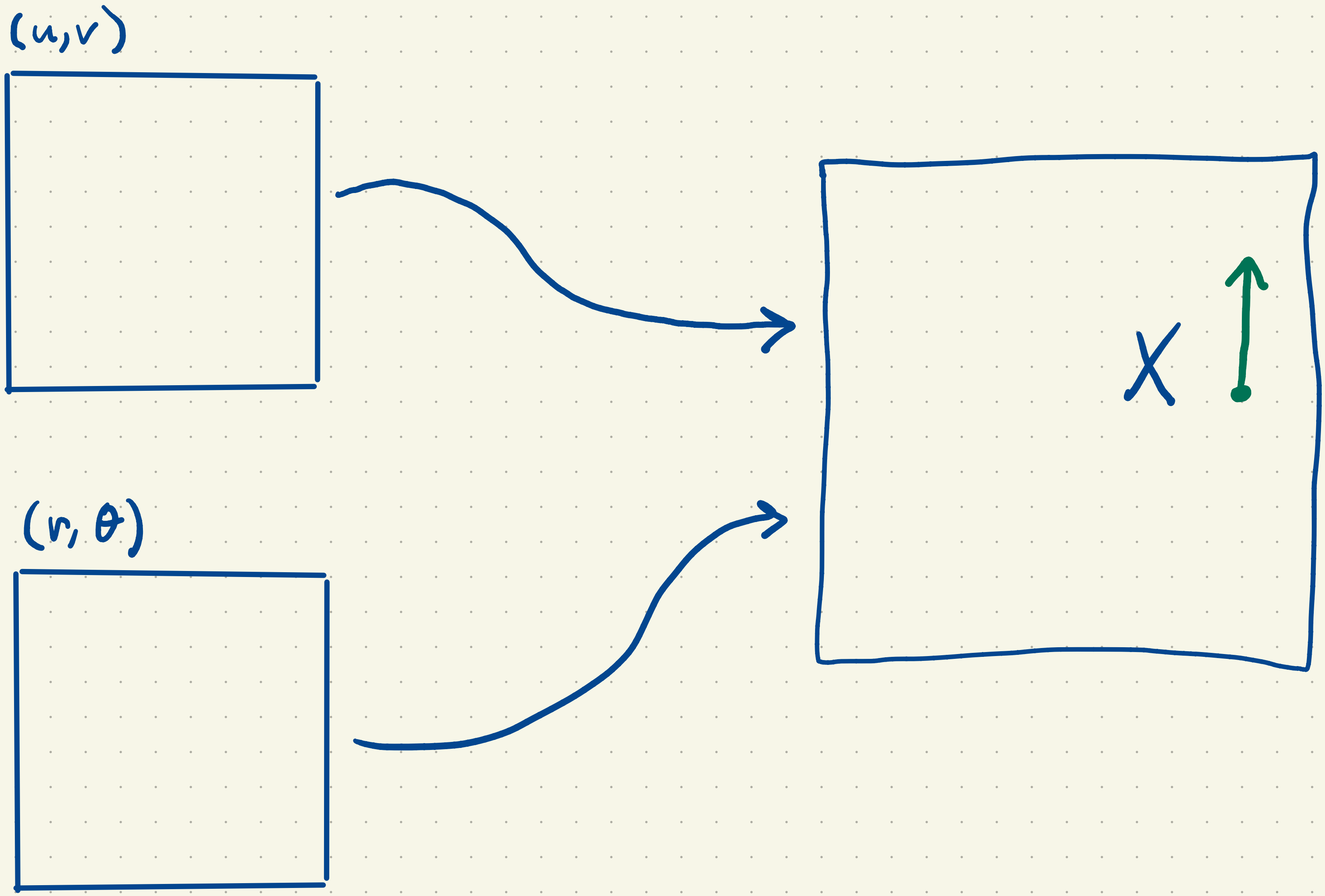
$$u(t) = U(r(t), \theta(t))$$

$$v(t) = V(r(t), \theta(t))$$

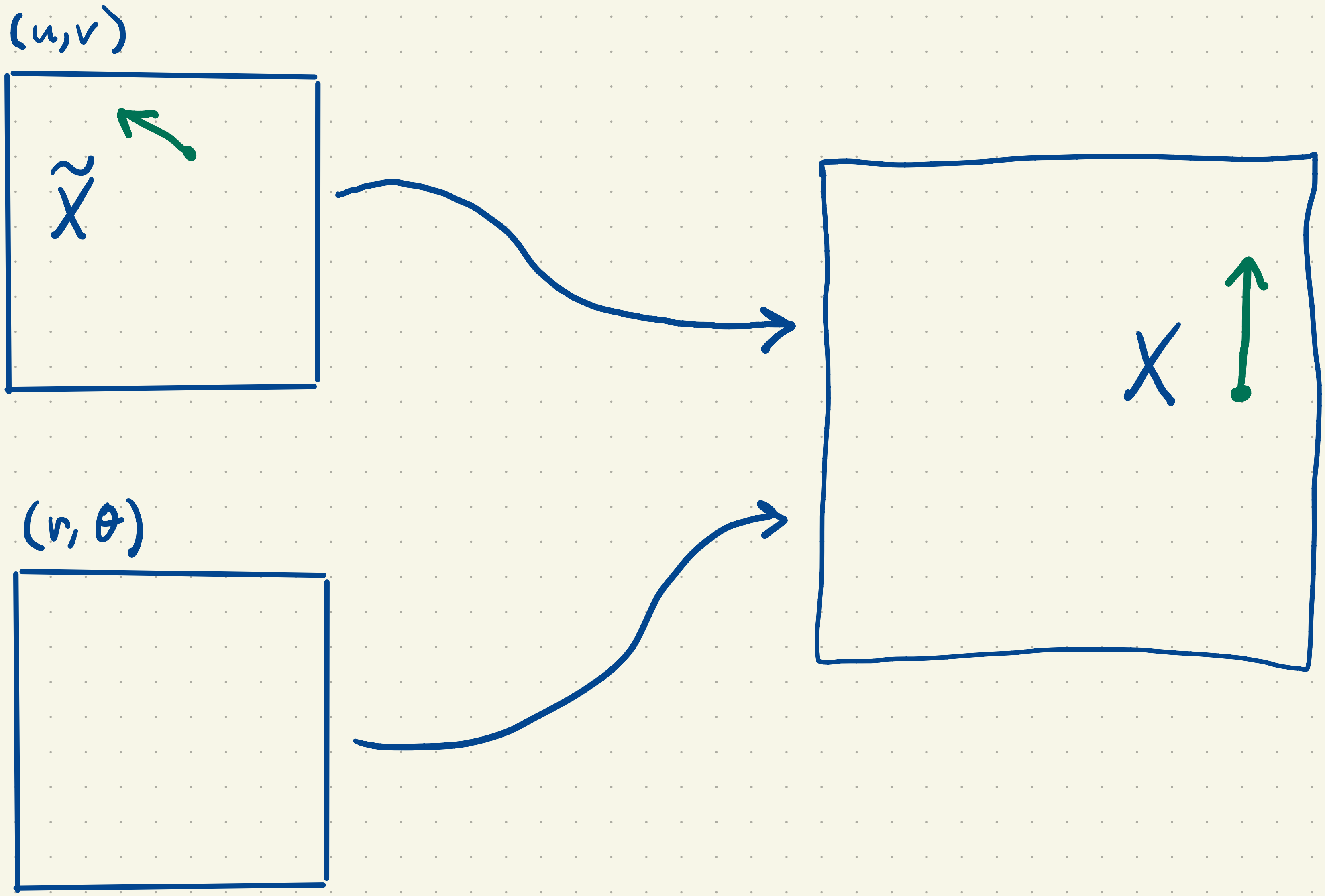
$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \partial U/\partial r & \partial U/\partial \theta \\ \partial V/\partial r & \partial V/\partial \theta \end{bmatrix} \begin{bmatrix} r' \\ \theta' \end{bmatrix}$$

J

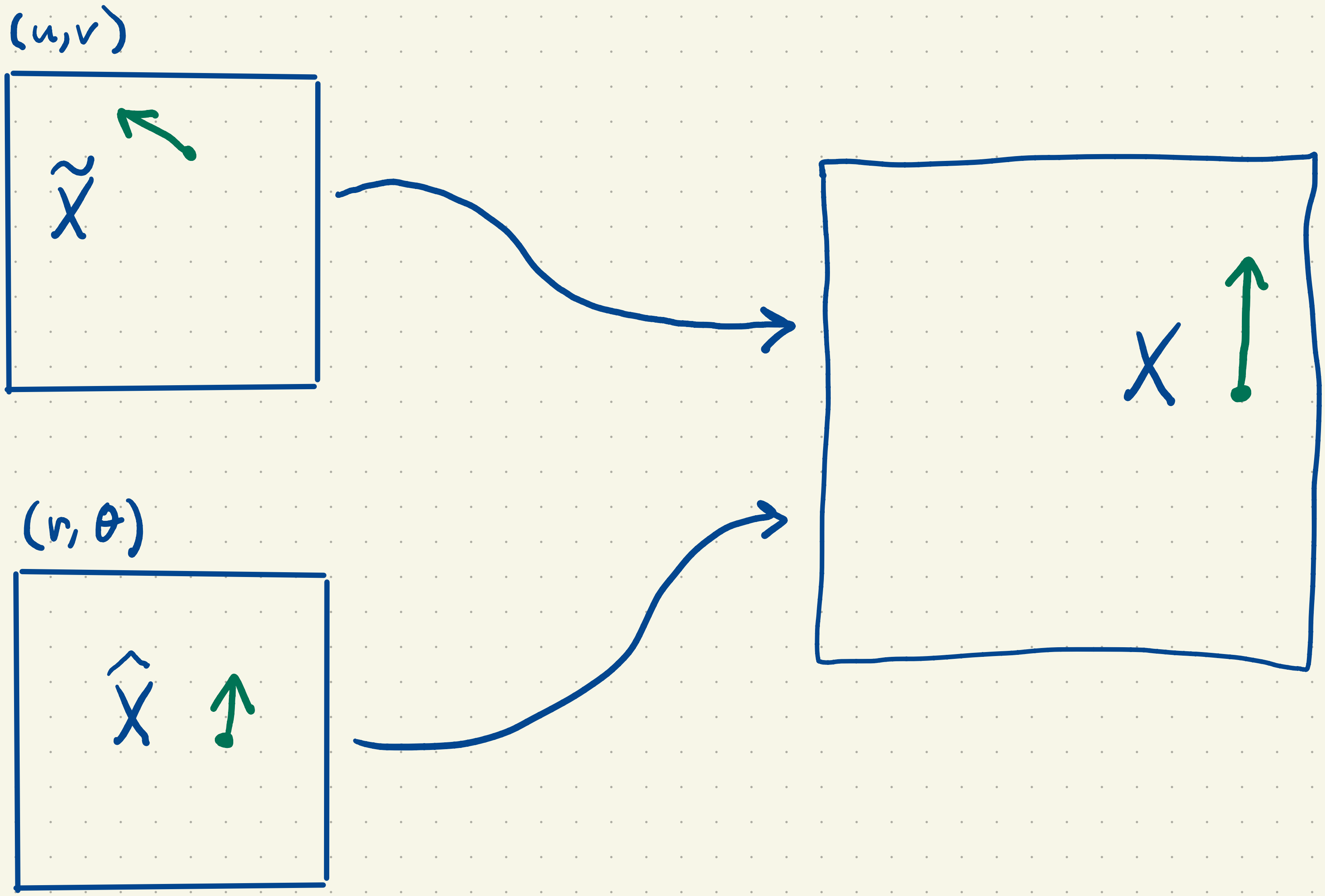
Tangent Vector = Infinitesimal Curve



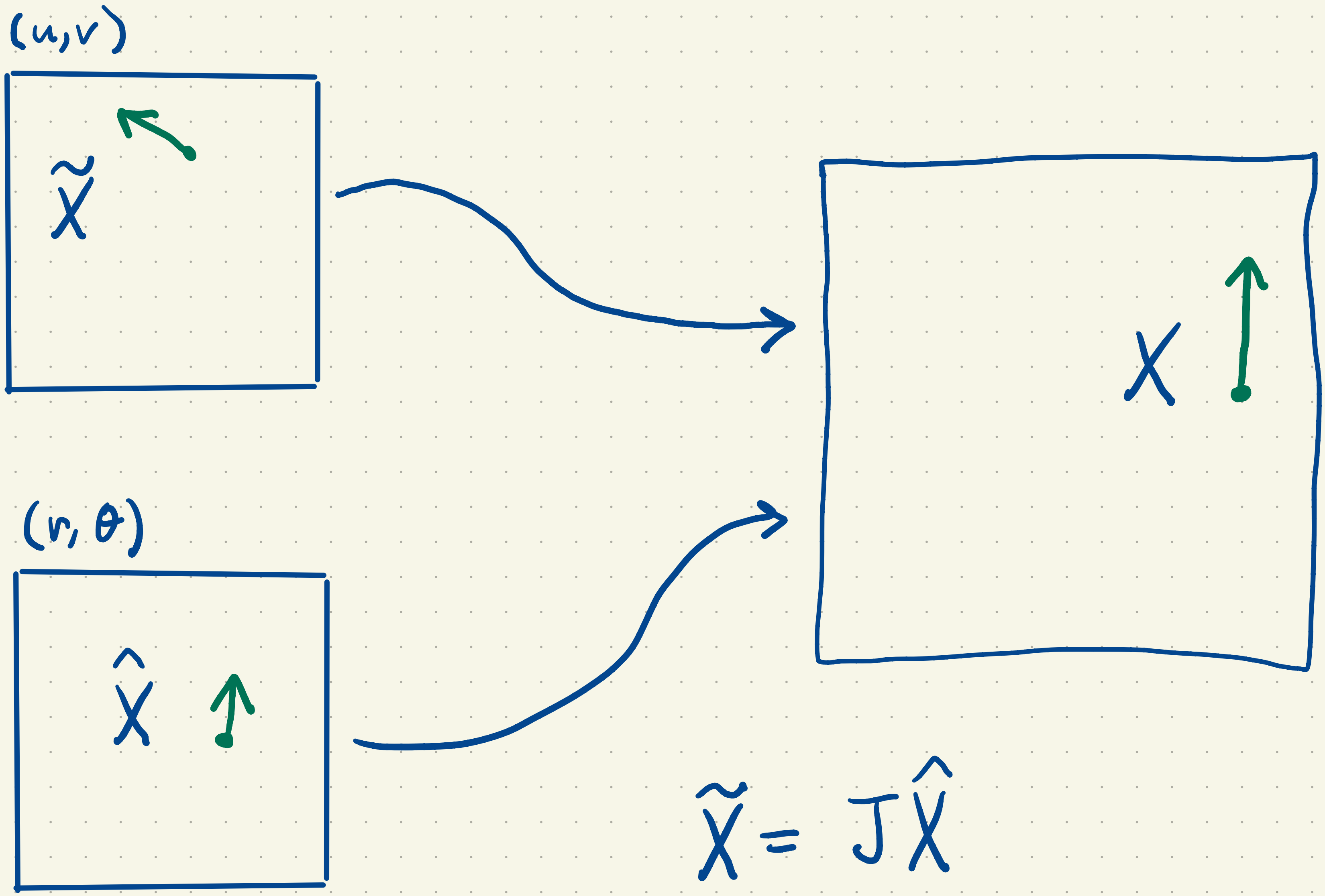
Tangent Vector = Infinitesimal Curve



Tangent Vector = Infinitesimal Curve

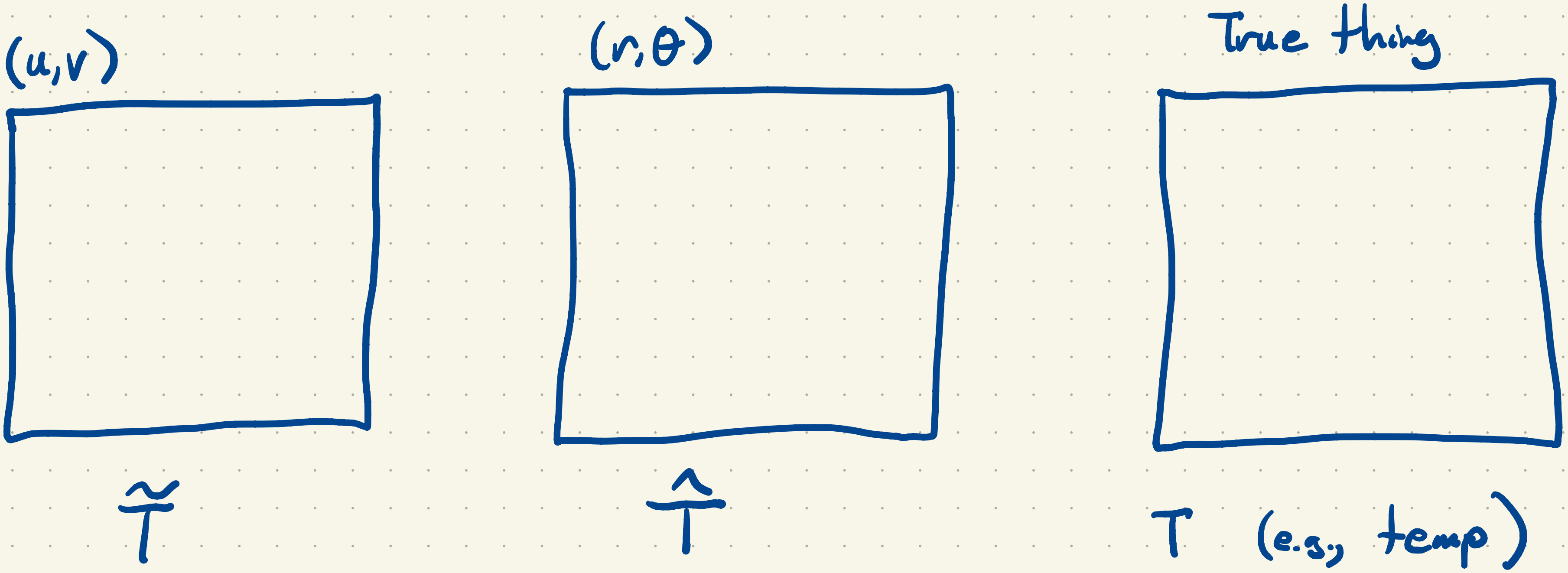


Tangent Vector = Infinitesimal Curve

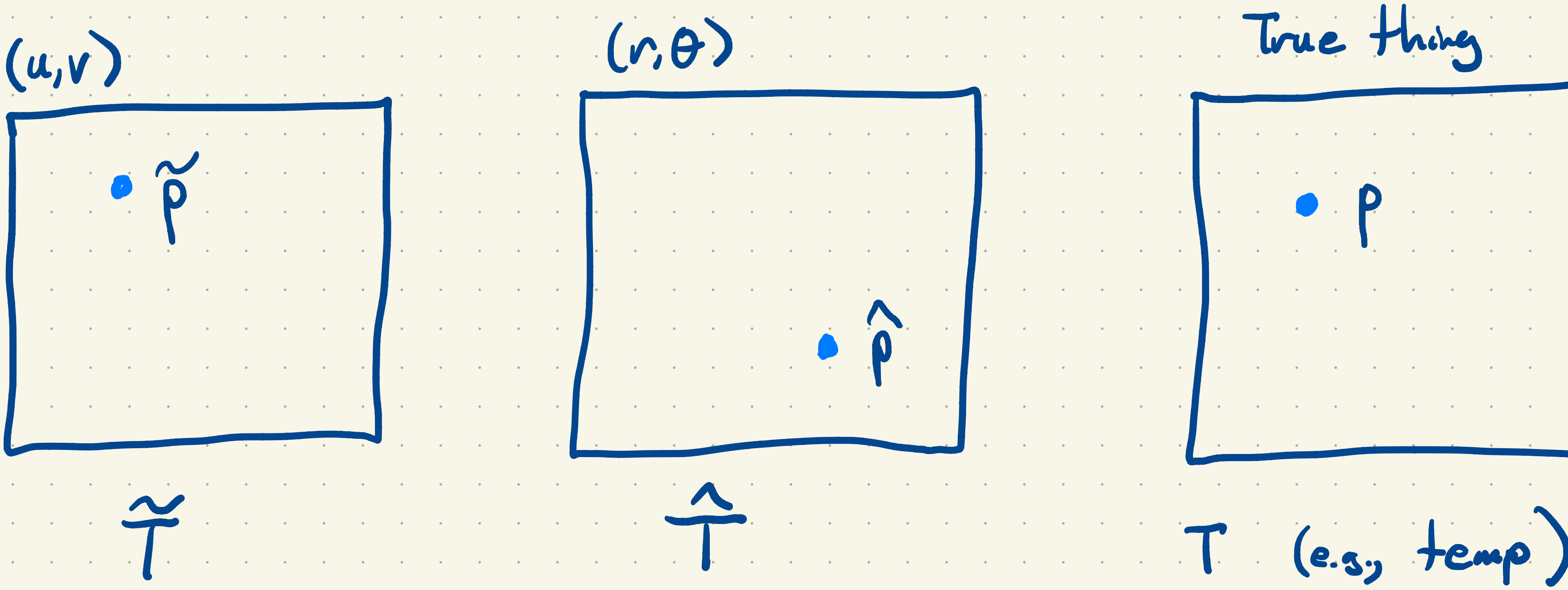


$$\tilde{X} = J\hat{X}$$

Covector = Infinitesimal Function

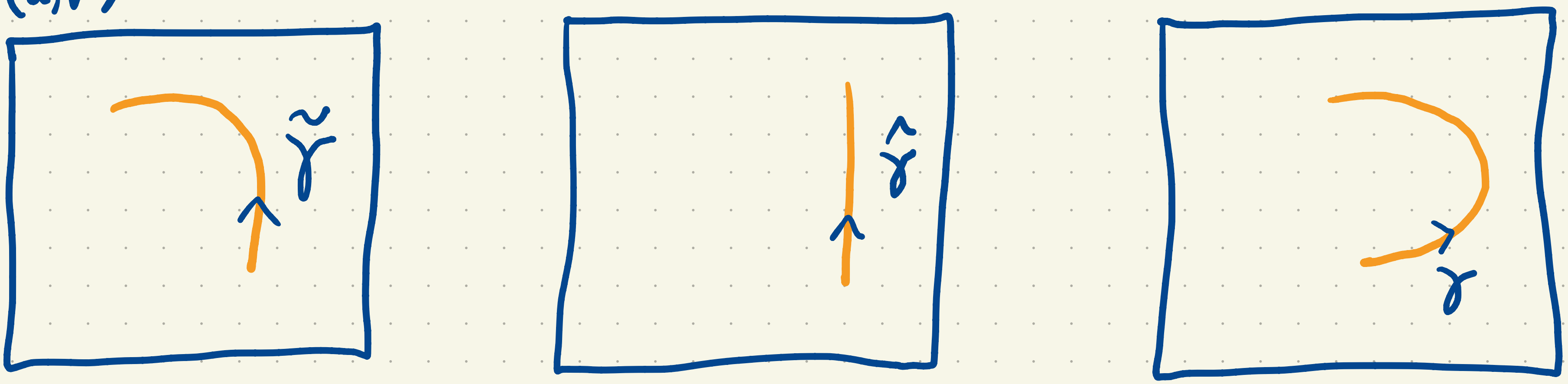


Covector = Infinitesimal Function



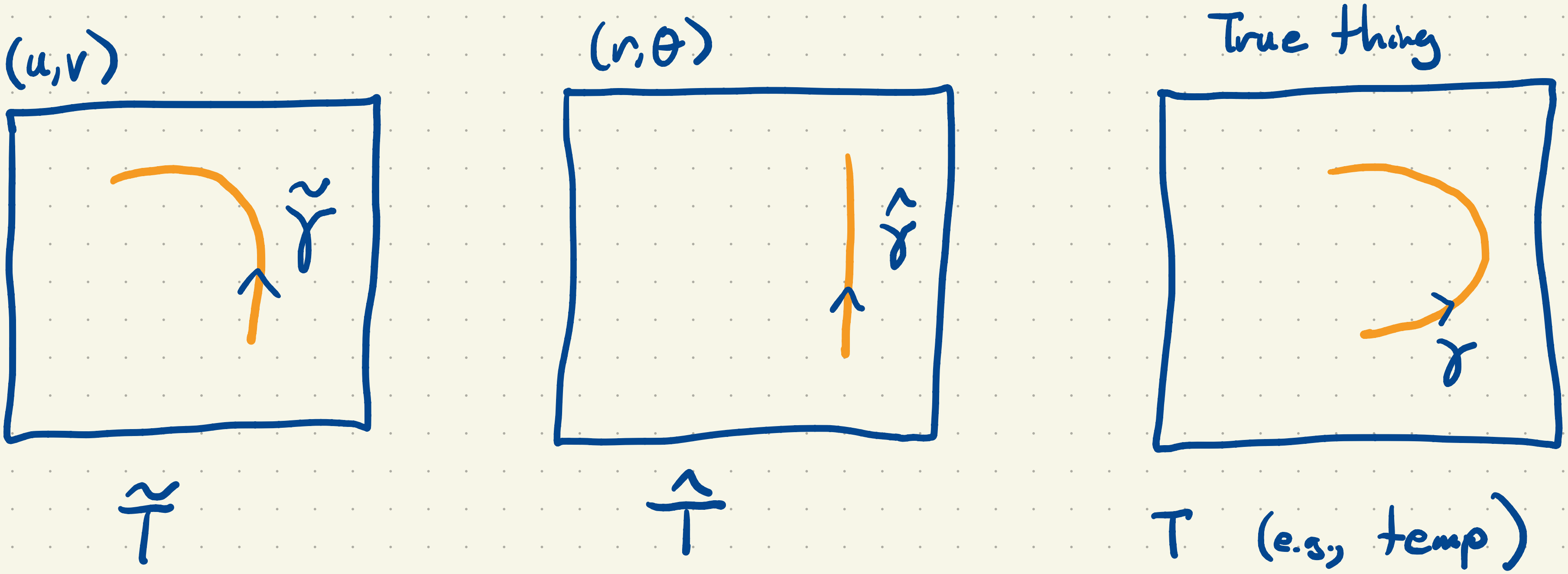
Covector = Infinitesimal Function

(u, v) (r, θ) True thing



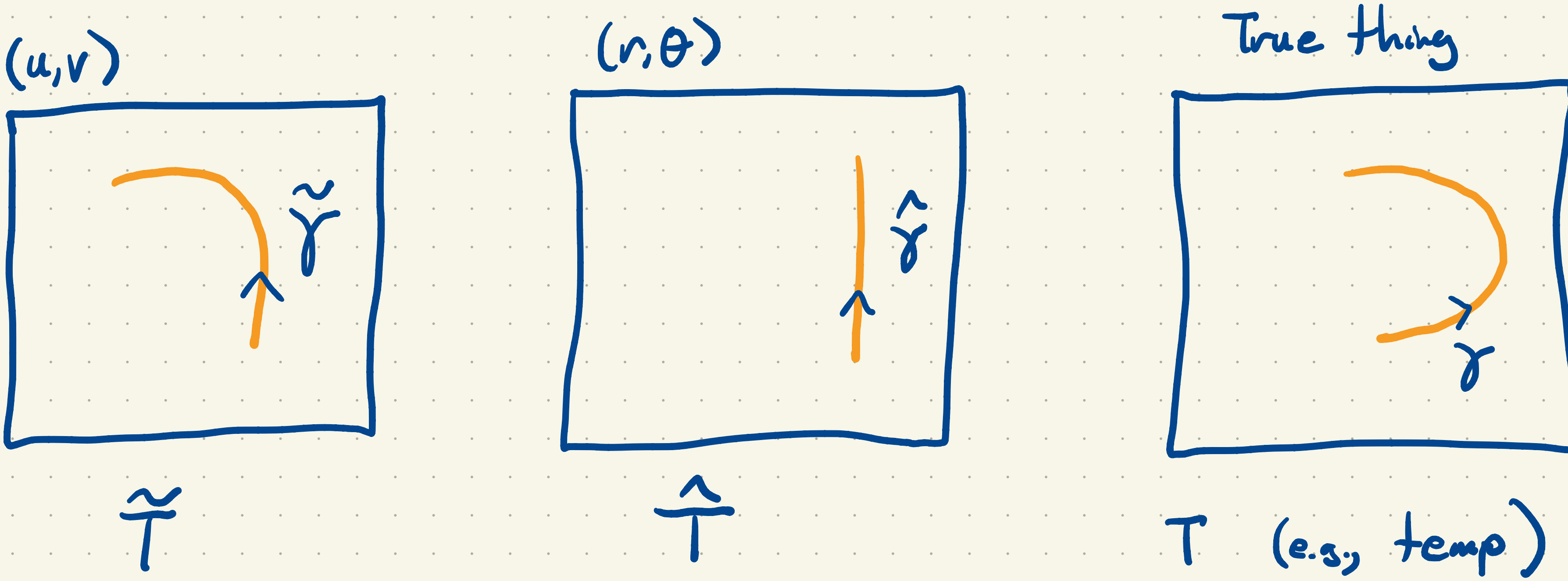
$\tilde{T}$ $\hat{T}$ T (e.g., temp)

Covector = Infinitesimal Function



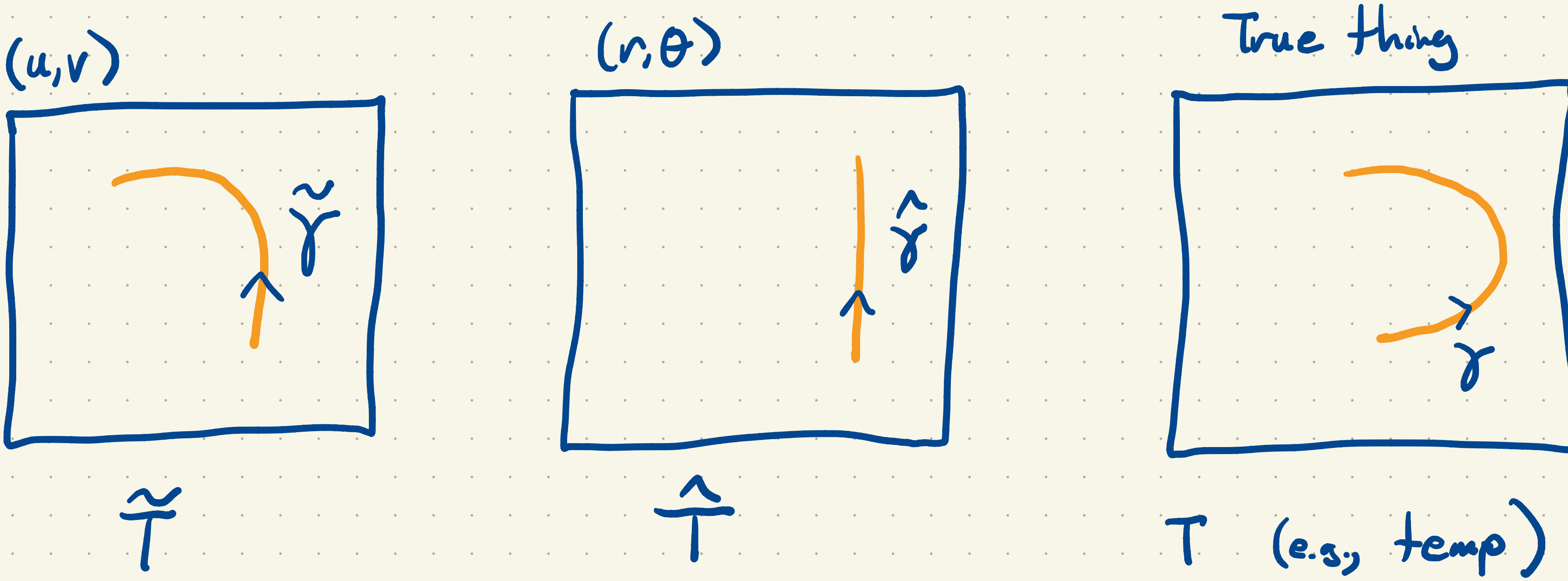
$$T(\gamma(t)) \;\longrightarrow\; \tilde{T}(\tilde{\gamma}(t))$$

Covector = Infinitesimal Function



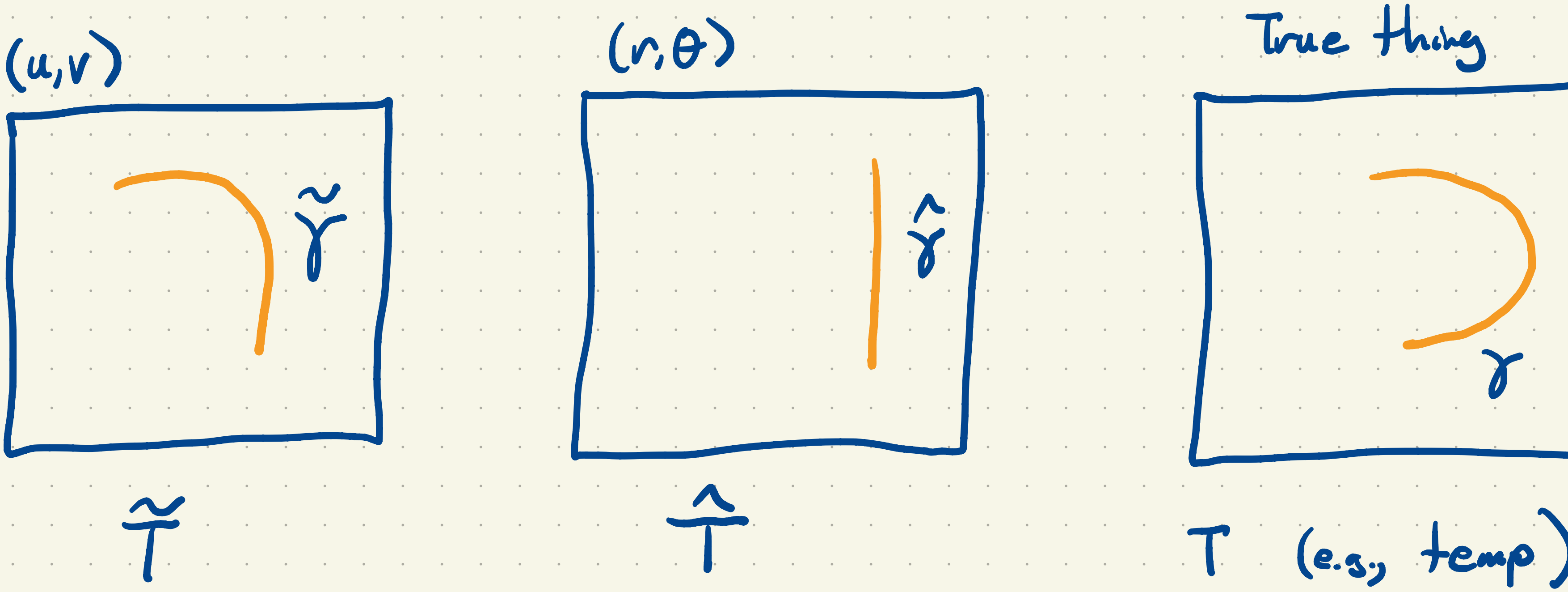
$$\begin{tikzcd}
& T(\gamma(t)) \arrow[dl] \arrow[dr] & \\
\tilde{T}(\tilde{\gamma}(t)) & & \hat{T}(\hat{\gamma}(t))
\end{tikzcd}$$

Covector = Infinitesimal Function



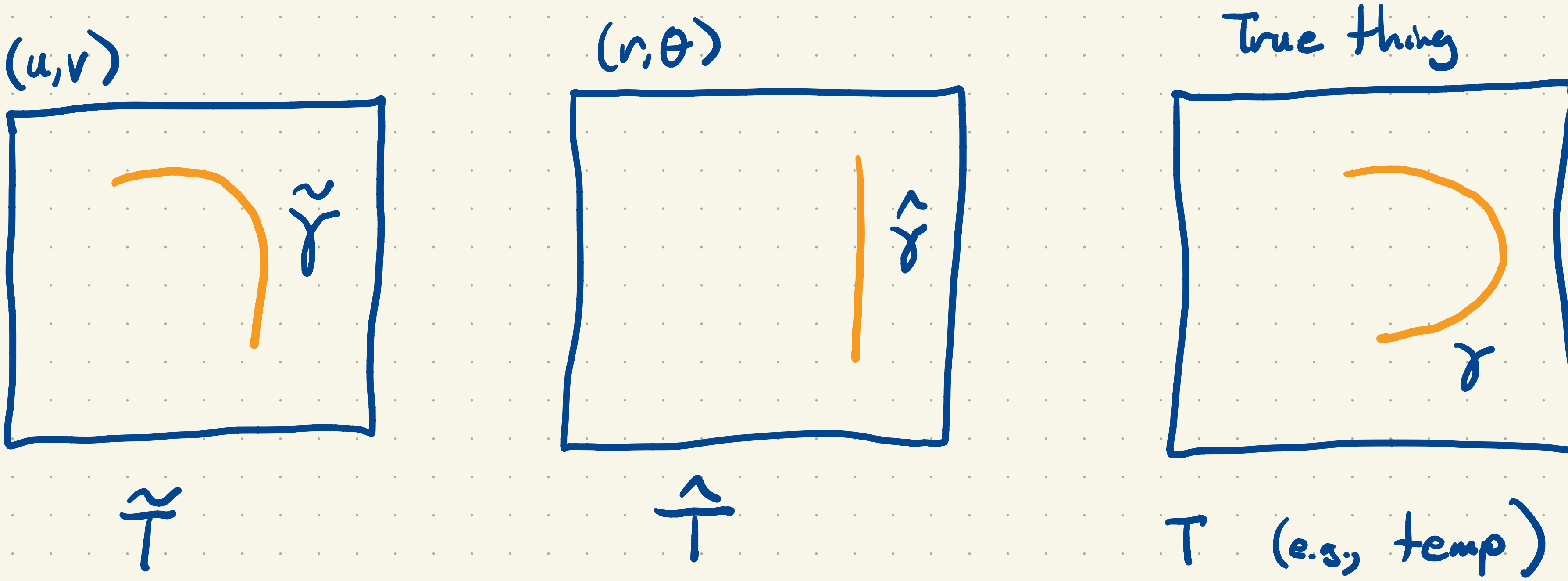
$$
\begin{tikzcd}
& T(\gamma(t)) \arrow[dl] \arrow[dr] & \\
\tilde{T}(\tilde{\gamma}(t)) & = & \hat{T}(\hat{\gamma}(t))
\end{tikzcd}
$$

Covector = Infinitesimal Function



$$\frac{d}{dt} T(\gamma(t))$$

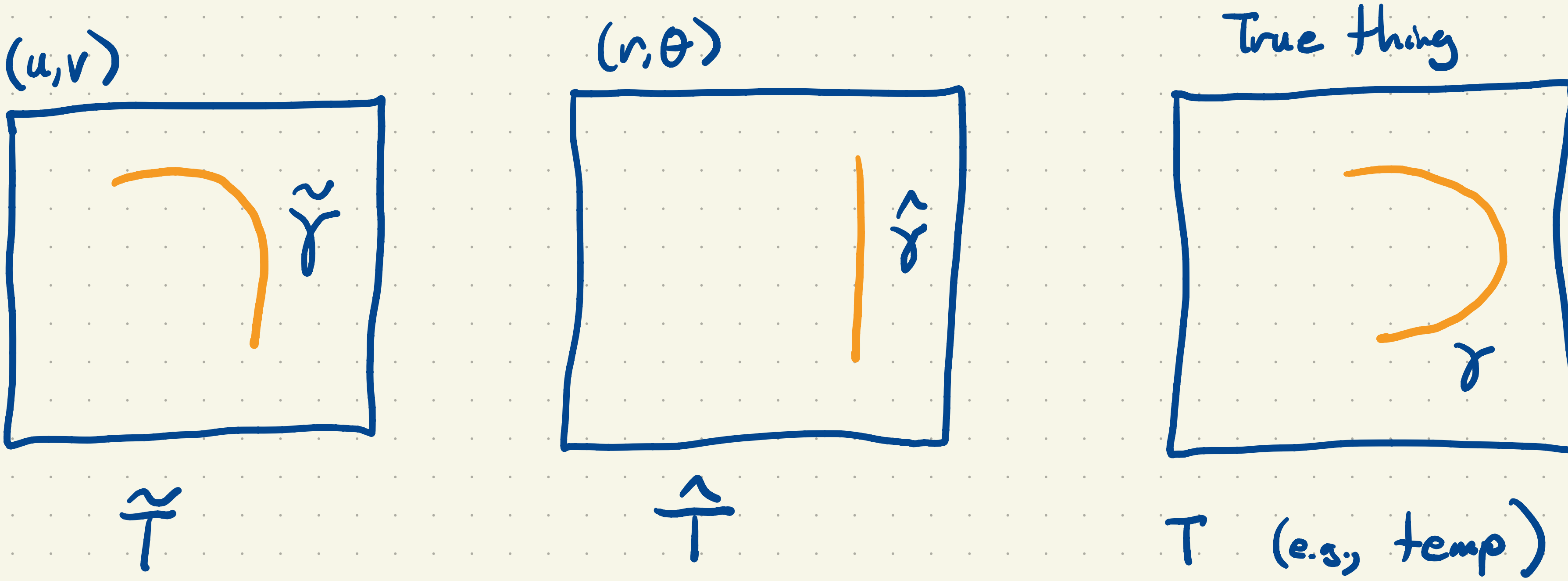
Covector = Infinitesimal Function



$$\frac{d}{dt} T(\gamma(t))$$

$$\frac{d}{dt} \tilde{T}(\tilde{\gamma}(t)) = \frac{d}{dt} \hat{T}(\hat{\gamma}(t))$$

Covector = Infinitesimal Function

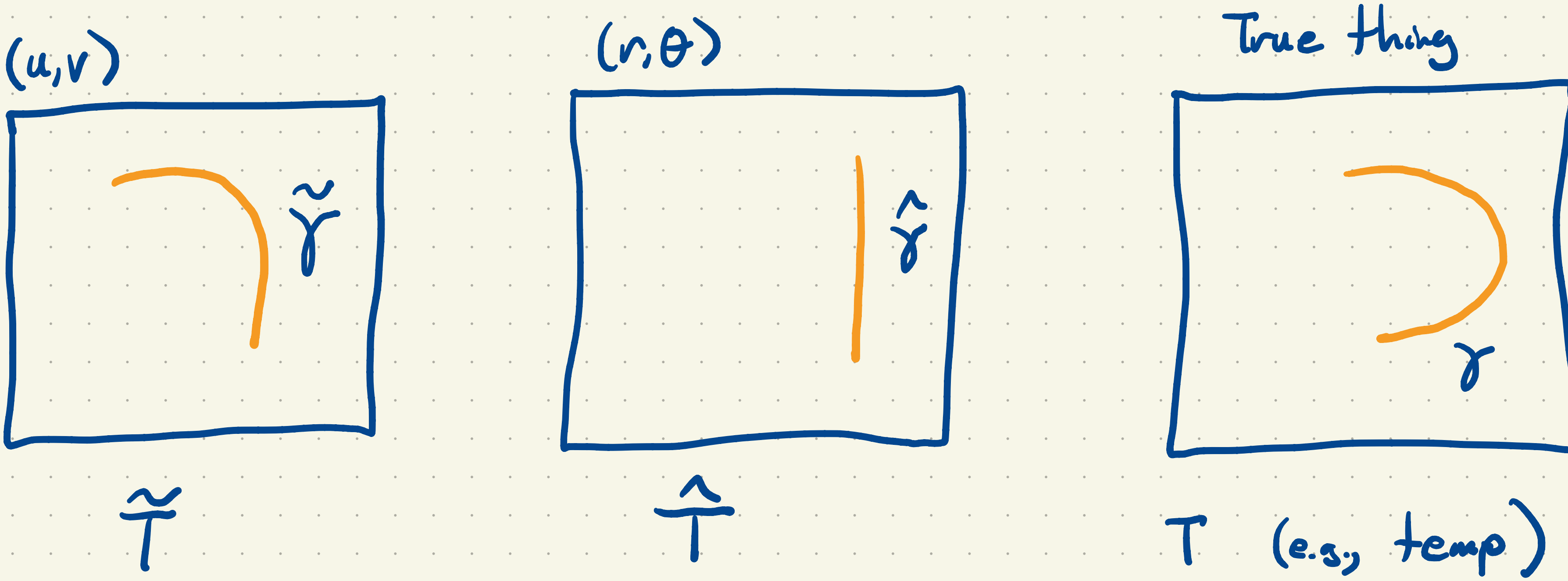


$$\left[\frac{\partial \tilde T}{\partial u}, \frac{\partial \tilde T}{\partial v}\right]\begin{bmatrix} u' \\ v' \end{bmatrix}$$

$$\uparrow$$

$$\frac{d}{dt}\tilde T(\tilde\gamma(t))$$

Covector = Infinitesimal Function

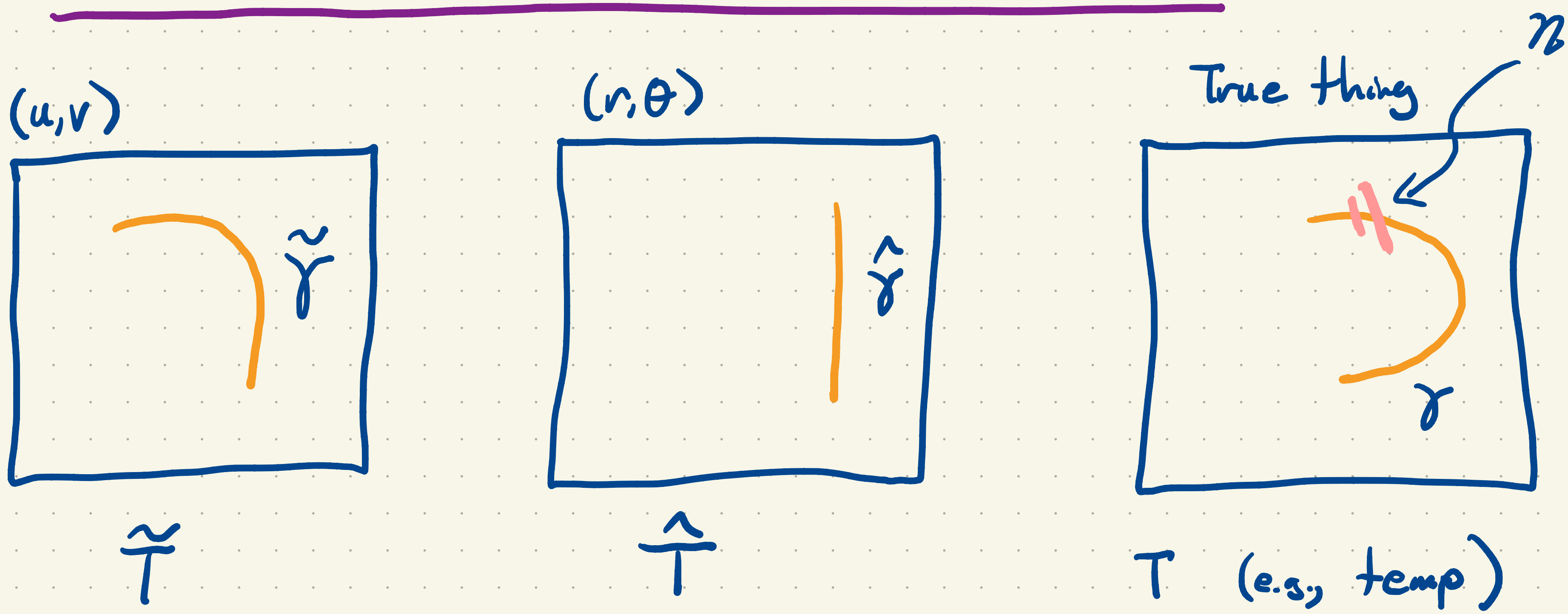


$$\begin{bmatrix} \frac{\partial \tilde{T}}{\partial u}, & \frac{\partial \tilde{T}}{\partial v} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{T}}{\partial r}, & \frac{\partial \hat{T}}{\partial v} \end{bmatrix} \begin{bmatrix} r' \\ \theta' \end{bmatrix}$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

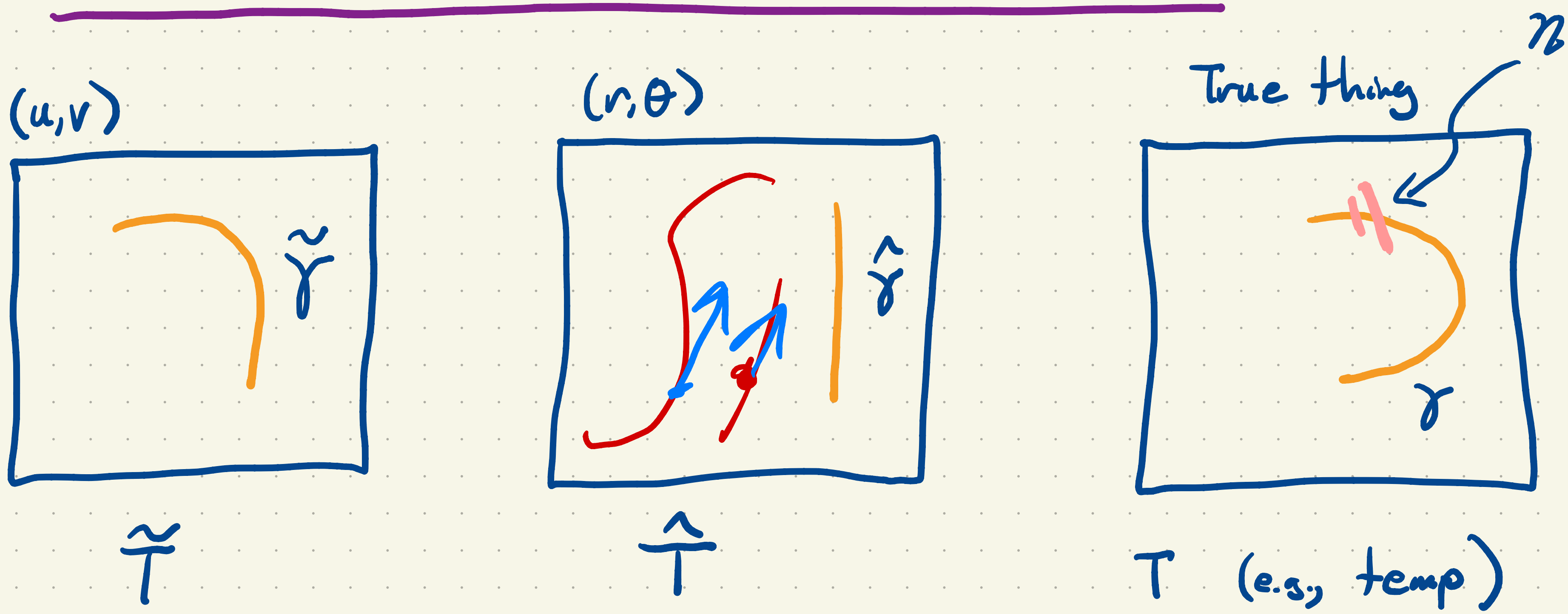
$$\frac{d}{dt}\tilde{T}(\tilde{\gamma}(t)) \qquad\qquad \frac{d}{dt}\hat{T}(\hat{\gamma}(t))$$

Covector = Infinitesimal Function



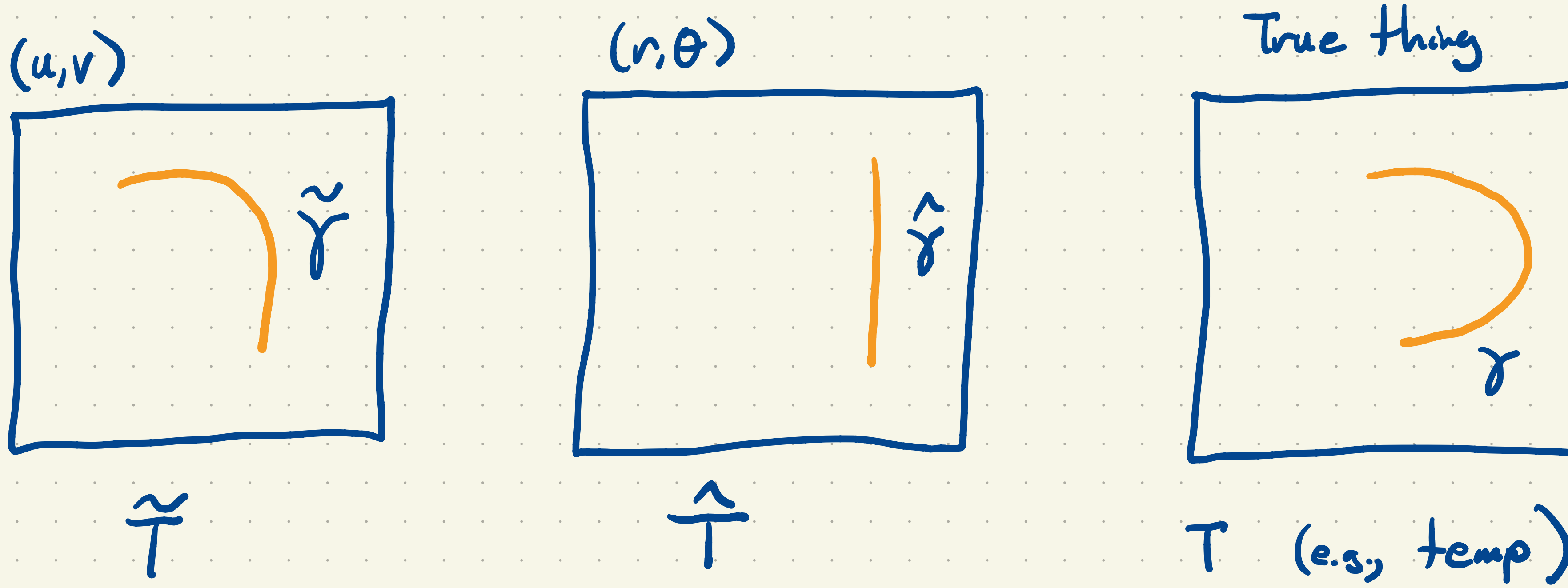
$$\underbrace{\left[\frac{\partial \tilde{T}}{\partial u},\ \frac{\partial \tilde{T}}{\partial v}\right]}_{\tilde{\eta}}\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\left[\frac{\partial \hat{T}}{\partial r},\ \frac{\partial \hat{T}}{\partial v}\right]}_{\hat{\eta}}\begin{bmatrix} r' \\ \theta' \end{bmatrix}$$

Covector = Infinitesimal Function



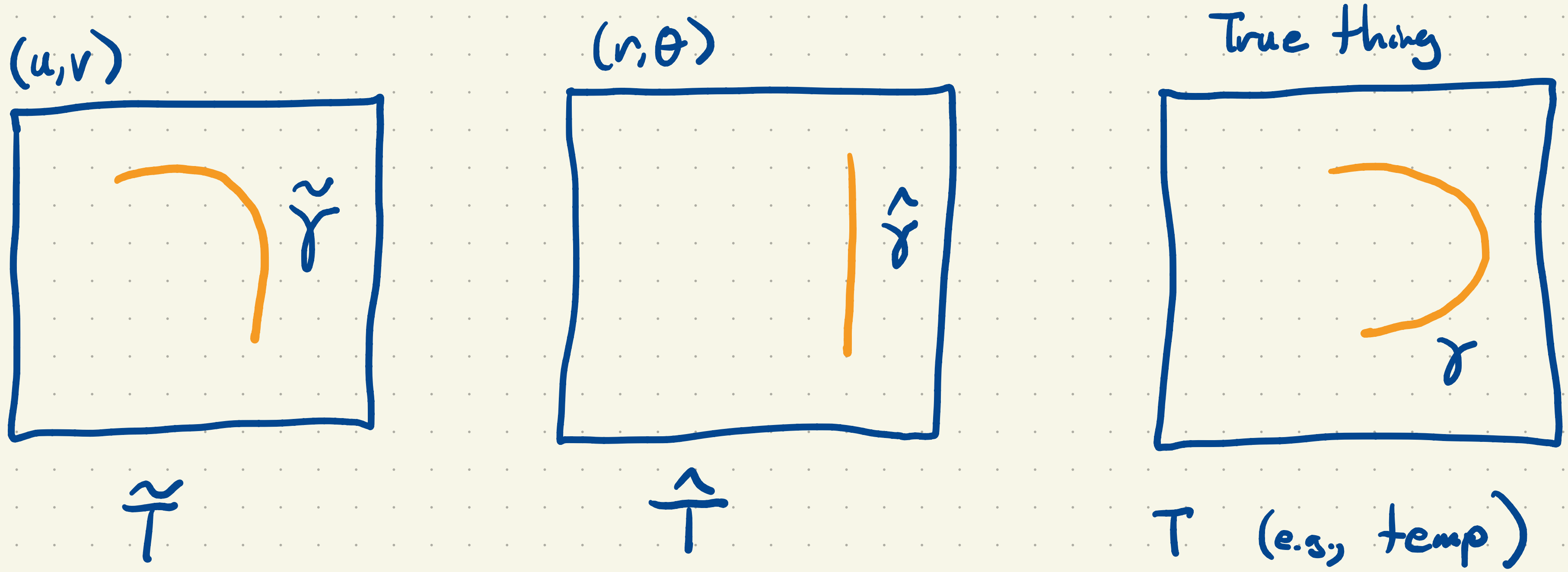
$$\underbrace{\left[\frac{\partial \tilde T}{\partial u}, \frac{\partial \tilde T}{\partial v}\right]}_{\tilde\eta}\begin{bmatrix}u'\\ v'\end{bmatrix} = \underbrace{\left[\frac{\partial \hat T}{\partial r}, \frac{\partial \hat T}{\partial v}\right]}_{\hat\eta}\begin{bmatrix}r'\\ \theta'\end{bmatrix}$$

Covector = Infinitesimal Function



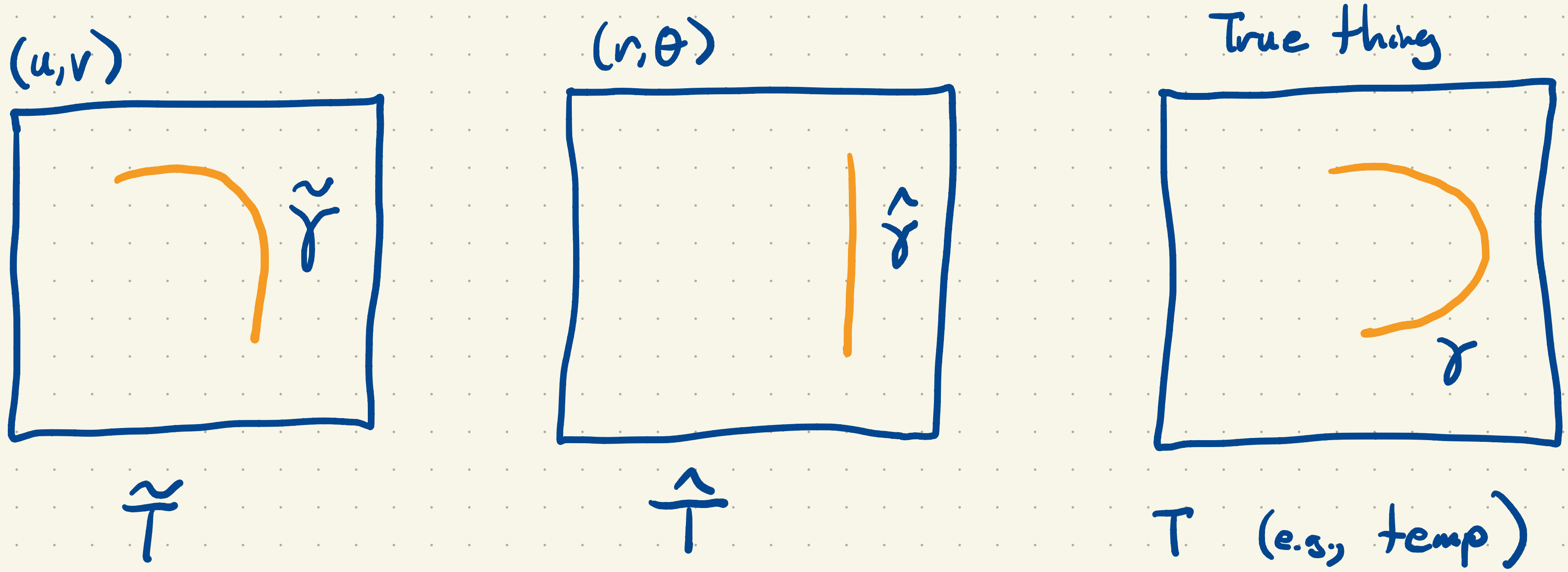
$$\left[\frac{\partial \tilde{T}}{\partial u}, \frac{\partial \tilde{T}}{\partial v}\right] J \begin{bmatrix} r' \\ \theta' \end{bmatrix} = \left[\frac{\partial \hat{T}}{\partial r}, \frac{\partial \hat{T}}{\partial v}\right] \begin{bmatrix} r' \\ \theta' \end{bmatrix}$$

Covector = Infinitesimal Function



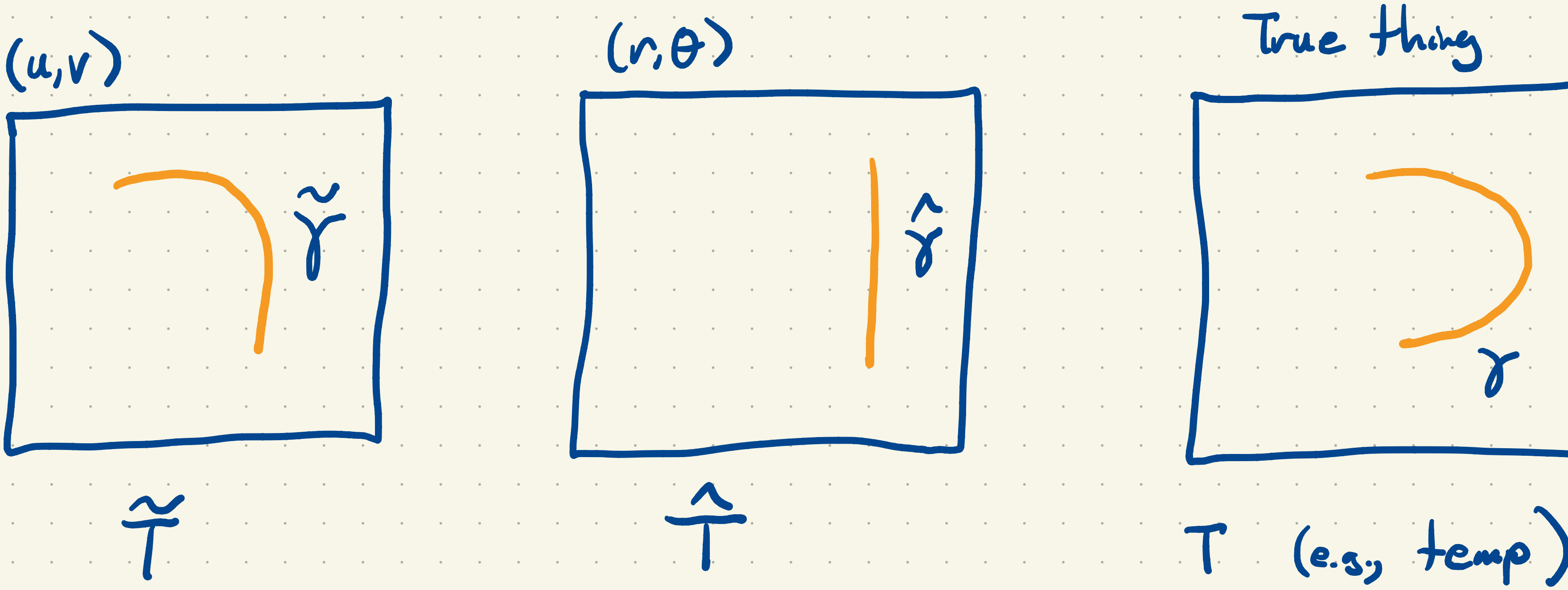
$$\left[\frac{\partial \tilde{T}}{\partial u}, \frac{\partial \tilde{T}}{\partial v} \right] J = \left[\frac{\partial \hat{T}}{\partial r}, \frac{\partial \hat{T}}{\partial \theta} \right]$$

Covector = Infinitesimal Function



$$\tilde{\eta} J = \hat{\eta}$$

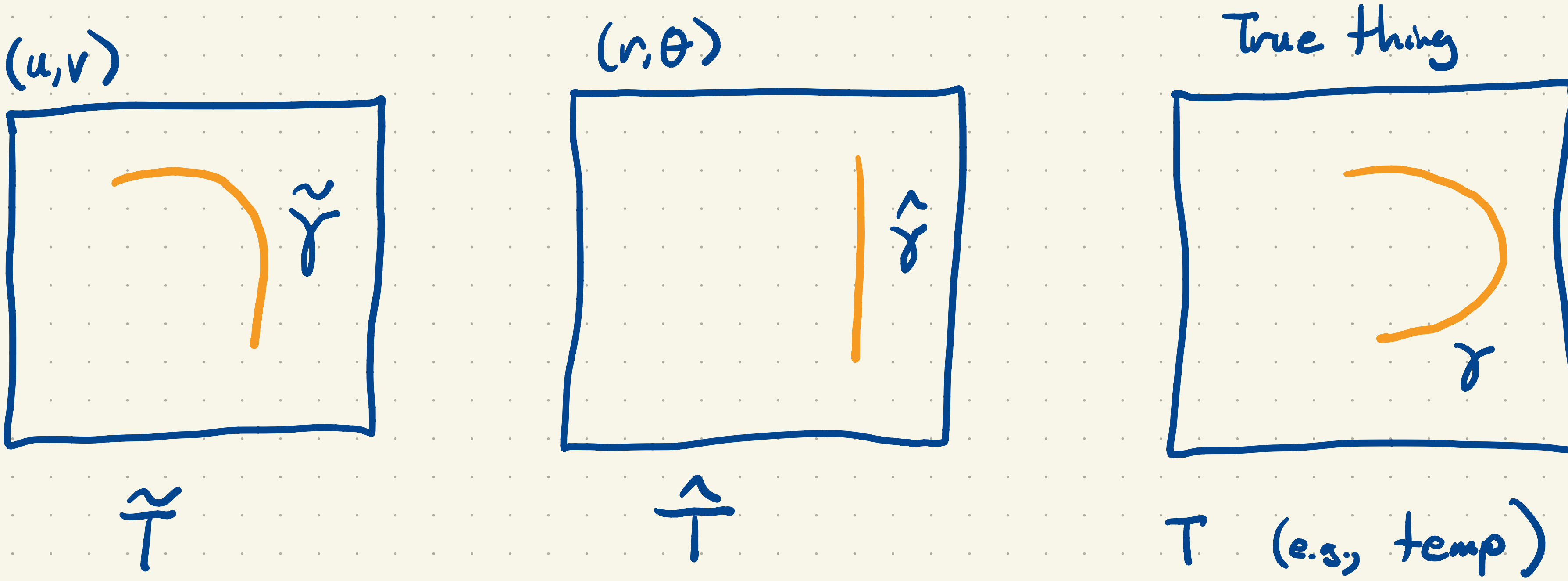
Covector = Infinitesimal Function



$$\tilde{\eta} J = \hat{\eta}$$

$$\tilde{X} = J \hat{X}$$

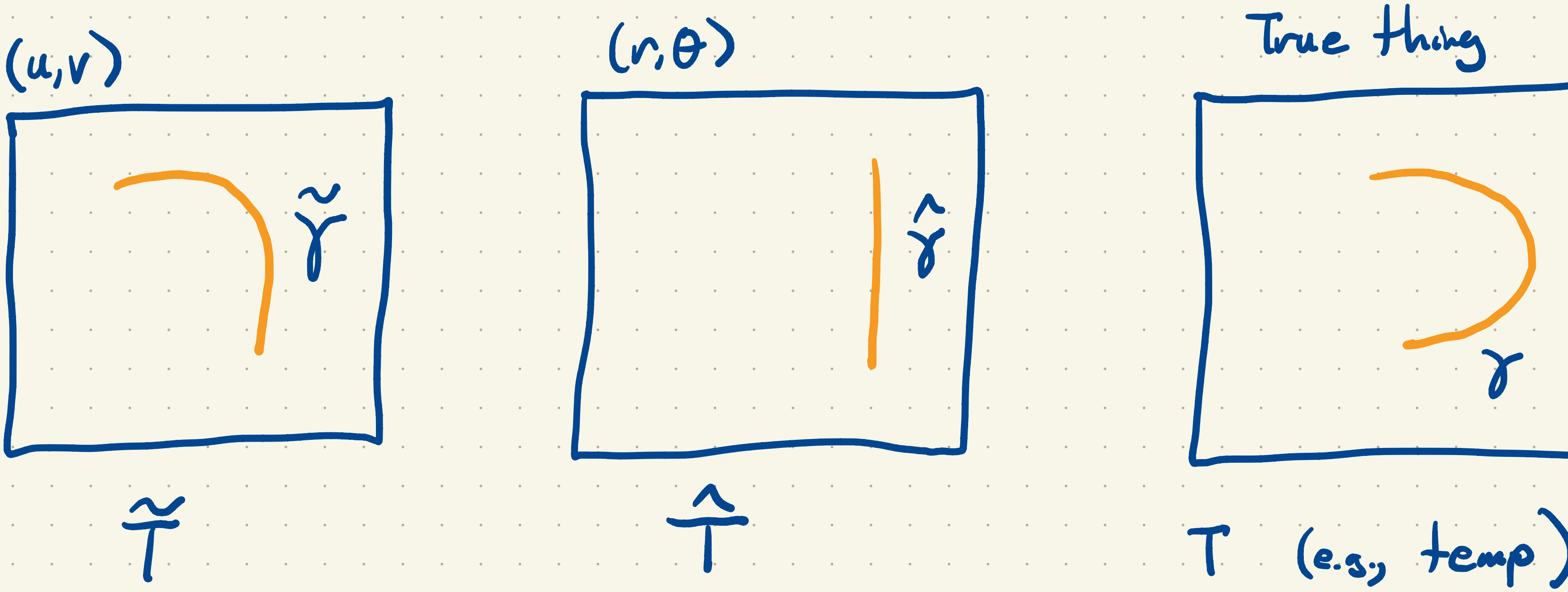
Covector = Infinitesimal Function



$$\tilde{n} = \hat{n} J^{-1}$$

$$\tilde{X} = J \hat{X}$$

Covector = Infinitesimal Function



$$\tilde{\eta}\,\tilde{X} = \hat{\eta}\, J^{-1} J \hat{X} = \hat{\eta}\,\hat{X}$$

Job of a Covector

Covectors eat vectors and give back numbers.

$$n[x] = \hat{n}\,\hat{x} = \tilde{n}\,\tilde{x}$$

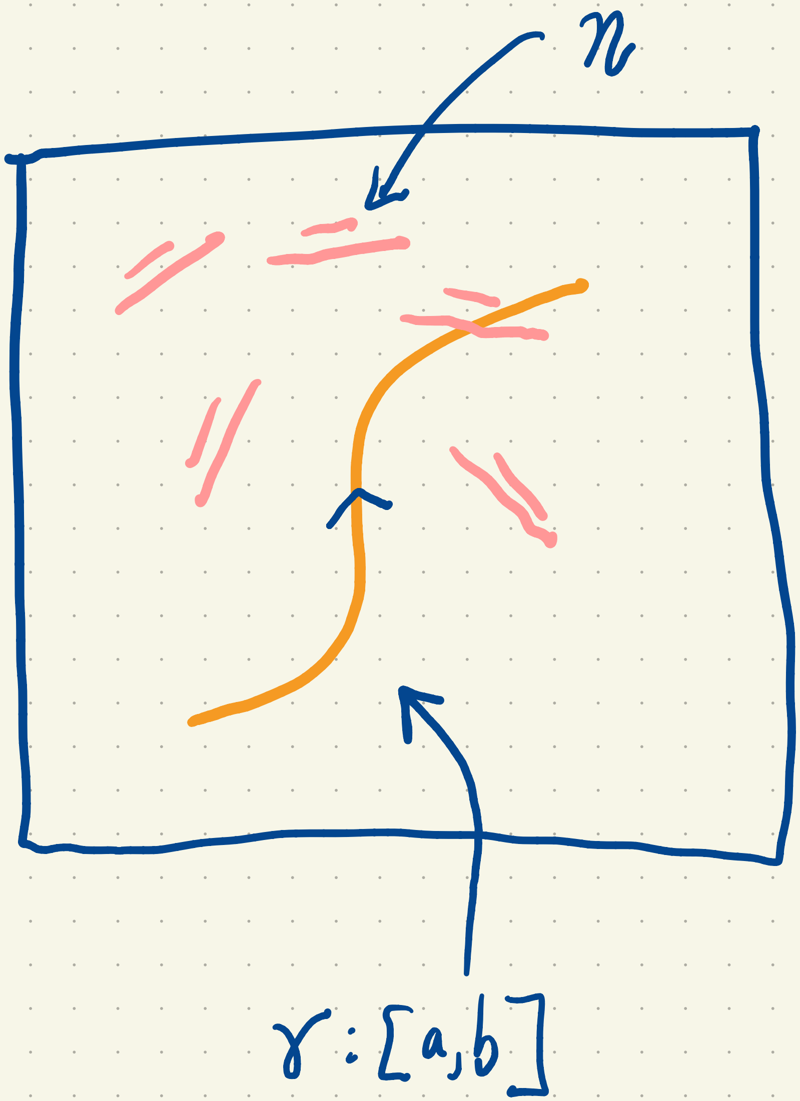
Job of a Covector

Covectors eat vectors and give back numbers.

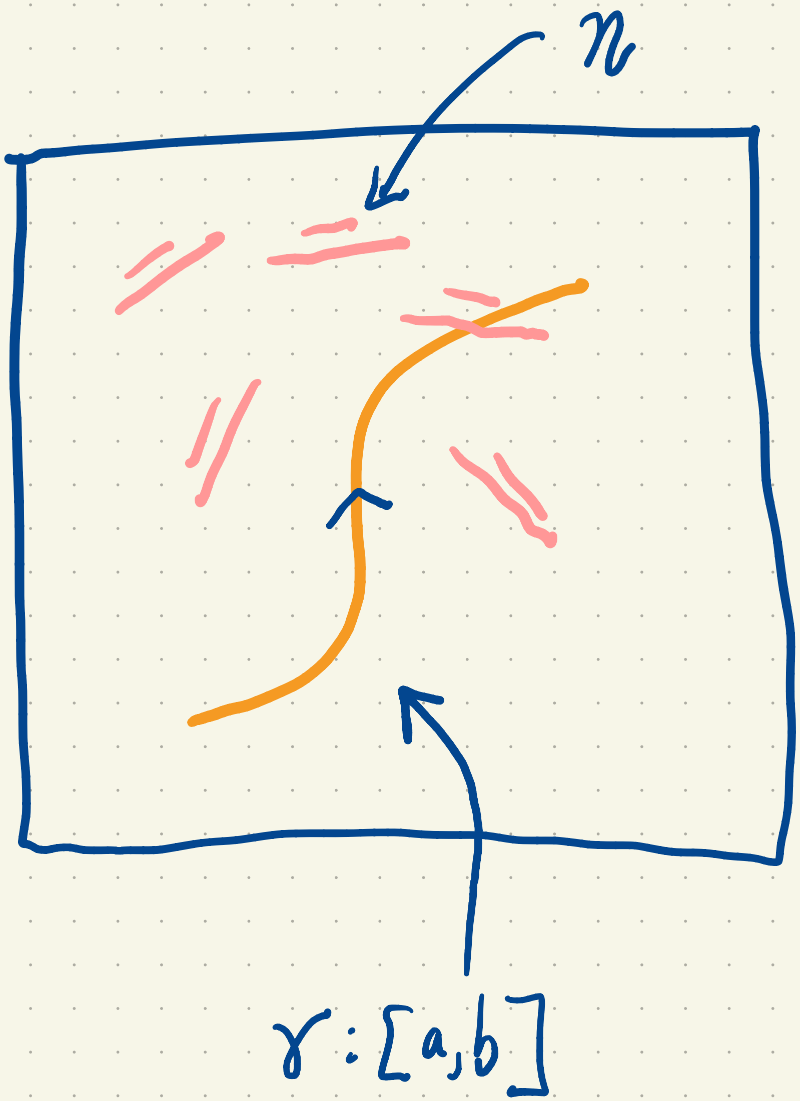
$$n[X] = \hat{n}\hat{X} = \tilde{n}\tilde{X}$$

"How fast is the infinitesimal function changing as I move with tangent vector X?"

Fields of Covectors Eat Curves

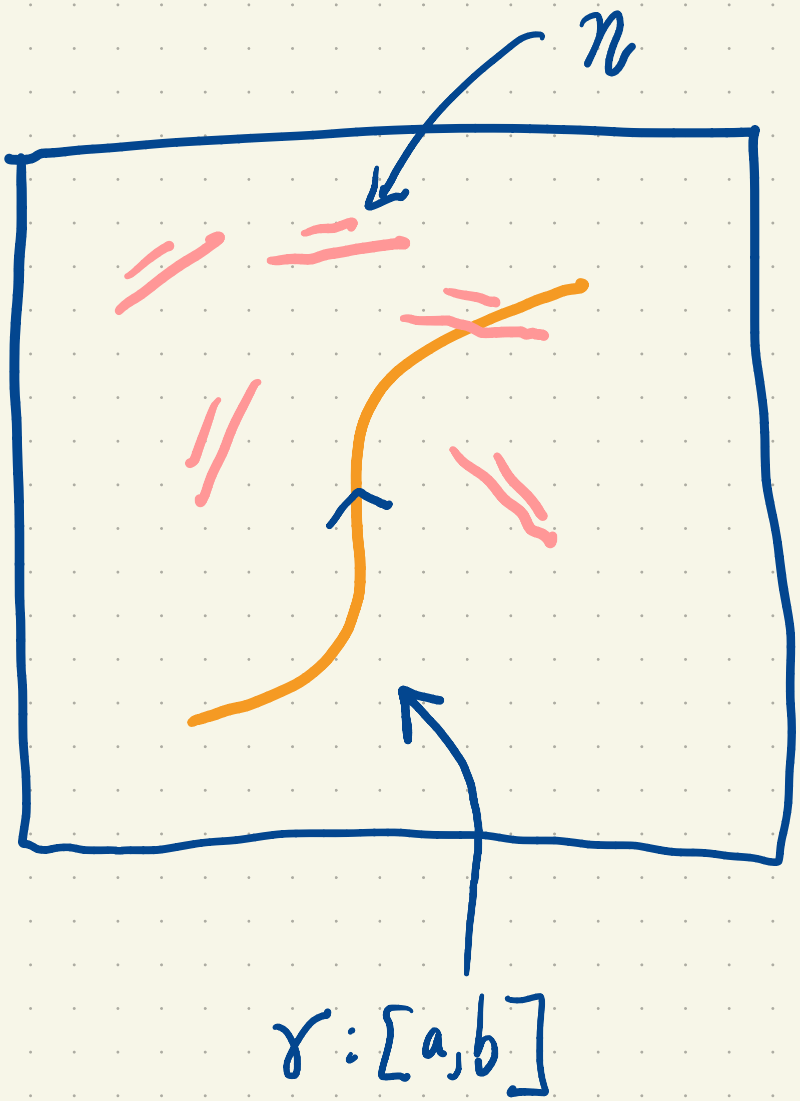


Fields of Covectors Eat Curves



$$\int_\gamma \eta = \int_a^b \eta[\gamma'(t)]\,dt$$

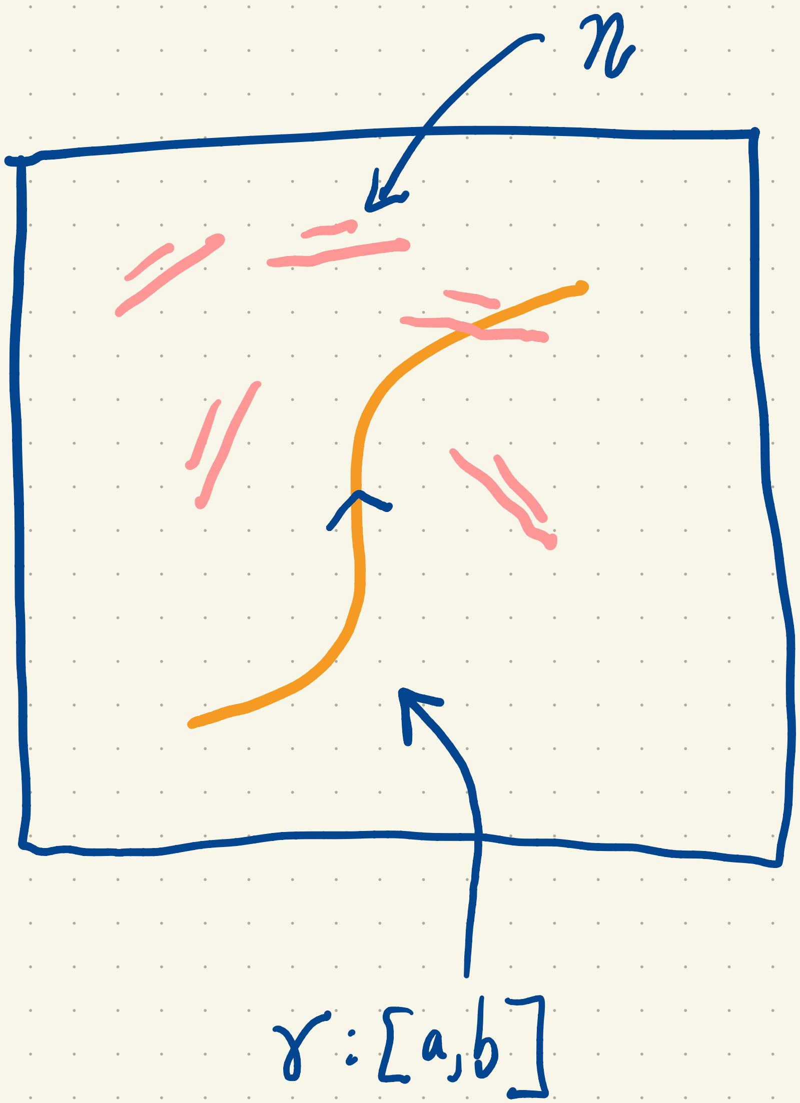
Fields of Covectors Eat Curves



$$\int_\gamma \eta = \int_a^b \eta[\gamma'(t)]\,dt$$

$$= \int_a^b \hat{\eta}\ \hat{\gamma}'\ dt$$

Fields of Covectors Eat Curves

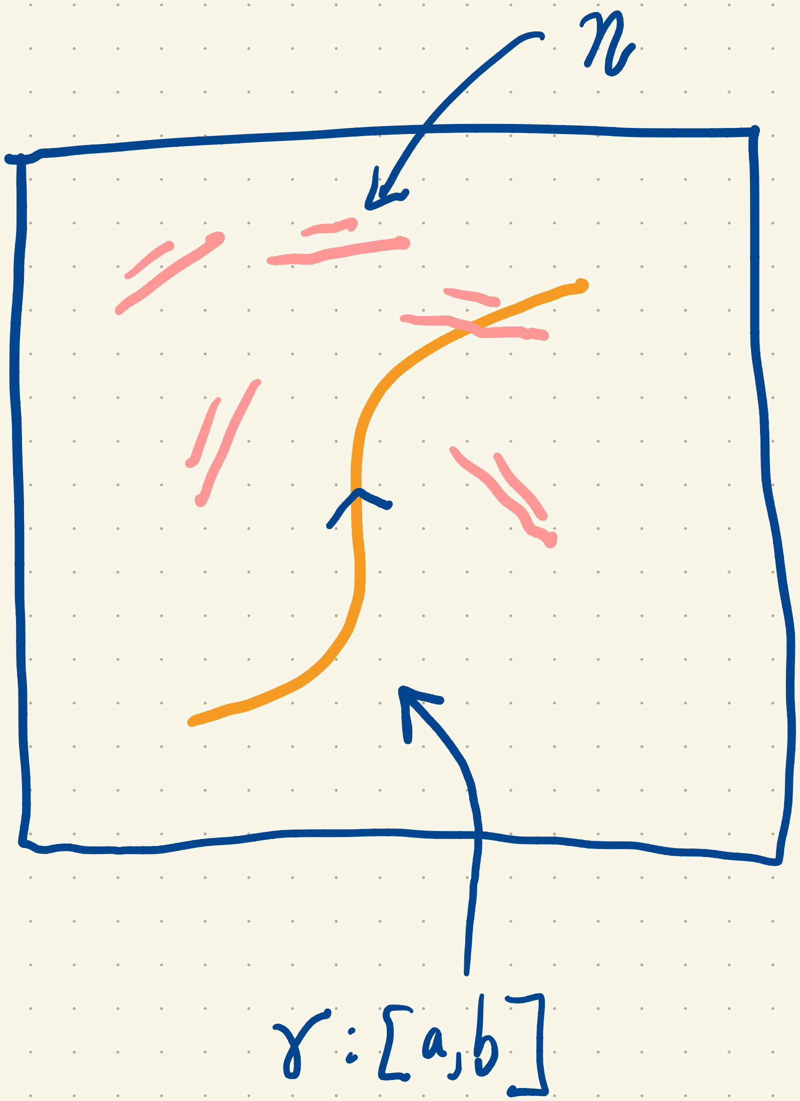


$$\int_\gamma \eta = \int_a^b \eta[\gamma'(t)]\,dt$$

$$= \int_a^b \hat{\eta}\,\hat{\gamma}'\,dt$$

$$= \int_a^b \tilde{\eta}\,\tilde{\gamma}'\,dt$$

Fields of Covectors Eat Curves



$$\int_\gamma \eta = \int_a^b \eta[\gamma'(t)]\,dt$$

$$= \int_a^b \hat{\eta}\,\hat{\gamma}'\,dt$$

$$= \int_a^b \tilde{\eta}\,\tilde{\gamma}'\,dt$$

Exterior Derivative (part 0)

$$T \xrightarrow{d} dT$$

function $\qquad\qquad\qquad$ covector

Exterior Derivative (part 0)

$$T \xrightarrow{d} dT$$

function $\qquad\qquad$ covector

$$dT[X] = \ ?$$

Exterior Derivative (part 0)

$$T \xrightarrow{d} dT$$

function $\qquad\qquad$ covector

$$dT[\gamma'] = \frac{d}{dt} T \circ \gamma$$

Exterior Derivative (part 0)

$$T \xrightarrow{d} dT$$

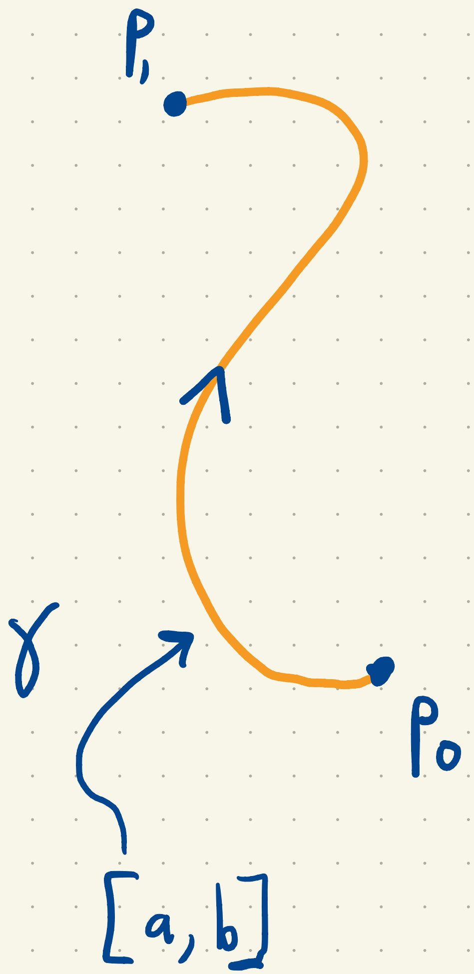
function covector

$$dT[\gamma'] = \frac{d}{dt} T \circ \gamma$$

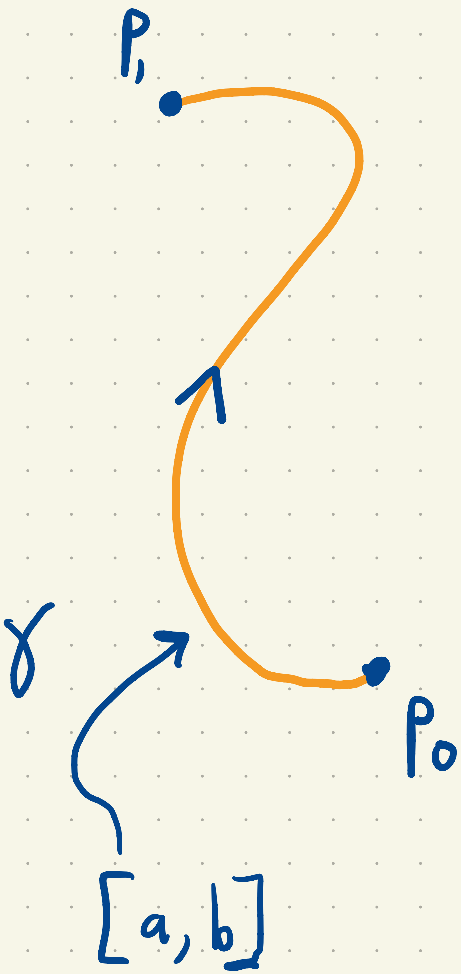
$$\left[\frac{\partial \tilde{T}}{\partial u}, \frac{\partial \tilde{T}}{\partial v} \right] \longleftrightarrow \left[\frac{\partial \hat{T}}{\partial r}, \frac{\partial \hat{T}}{\partial \theta} \right]$$

$$\widetilde{dT} \qquad \widehat{dT}$$

Exterior Derivative (part 0)

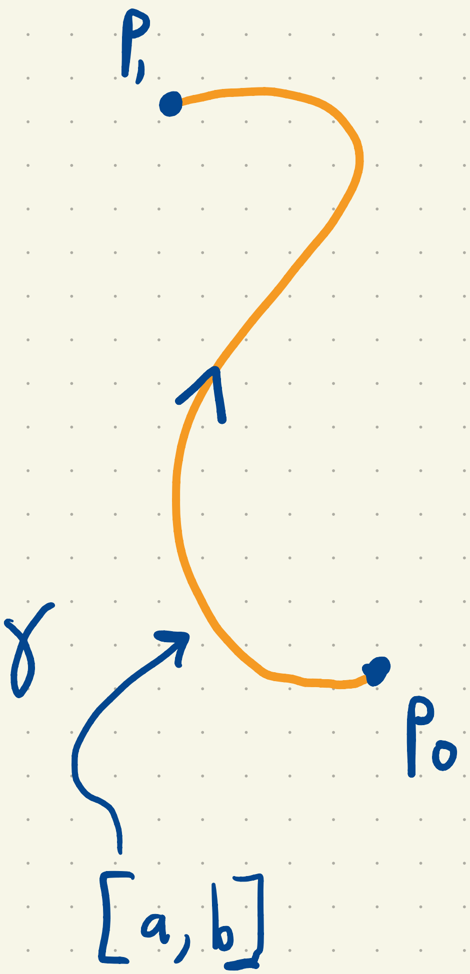


Exterior Derivative (part 0)



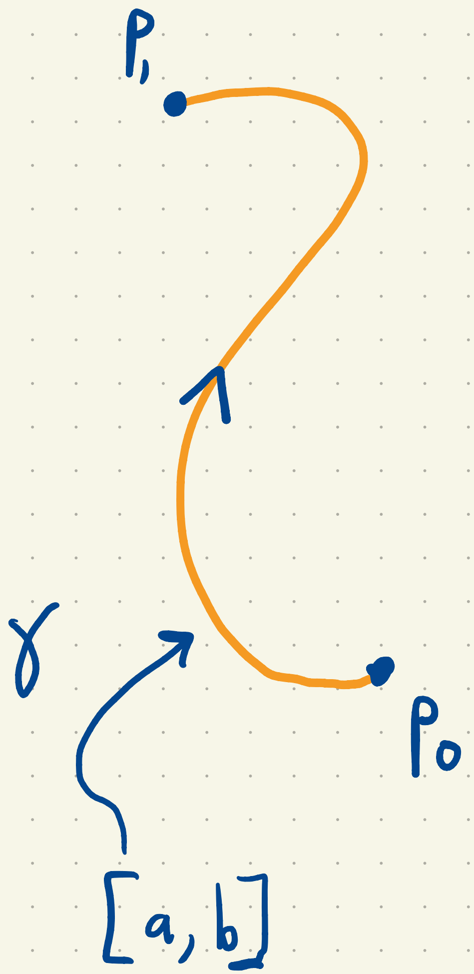
$$\int_\gamma dT$$

Exterior Derivative (part 0)



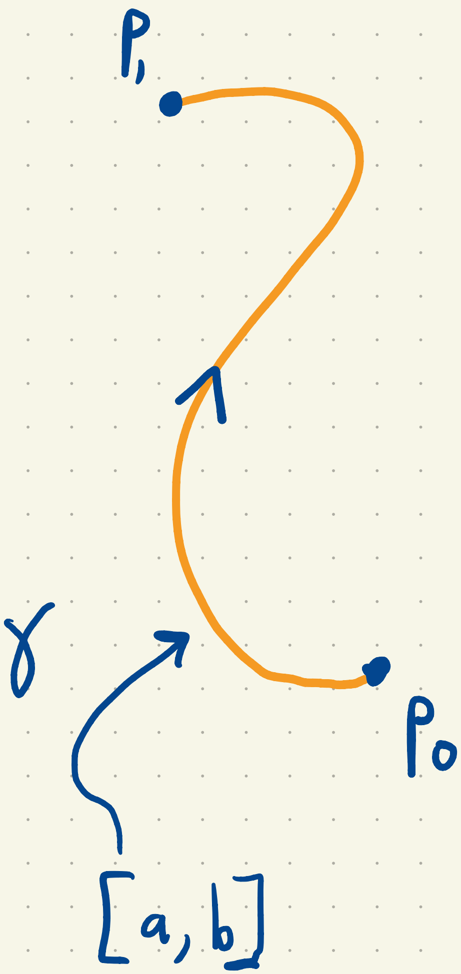
$$\int_\gamma dT = \int_a^b dT(\gamma'(t))\,dt$$

Exterior Derivative (part 0)



$$\int_\gamma dT = \int_a^b \frac{d}{dt} T(\gamma(t))\, dt$$

Exterior Derivative (part 0)



$$\int_\gamma dT = \int_a^b \frac{d}{dt} T(\gamma(t))\,dt$$

$$= T(\gamma(b)) - T(\gamma(a))$$

$$\boxed{\underline{\text{FTC}}: \int_a^b f'(s)\,ds = f(b) - f(a)}$$

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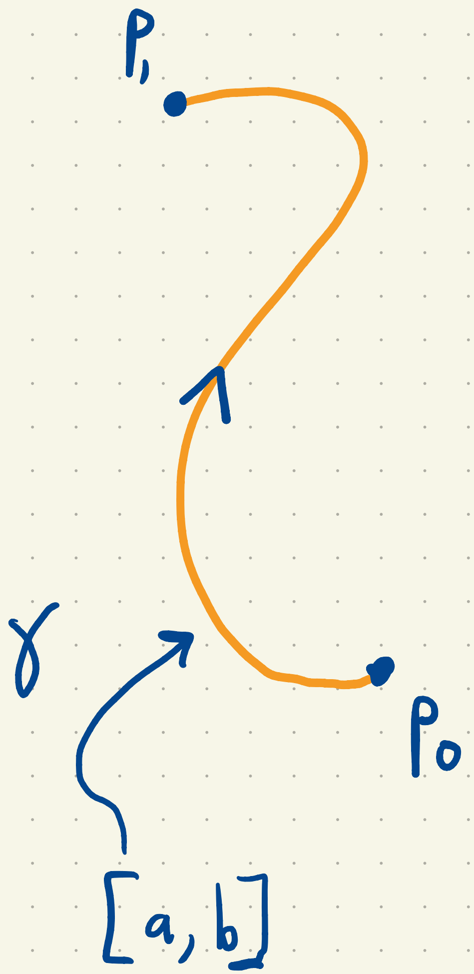


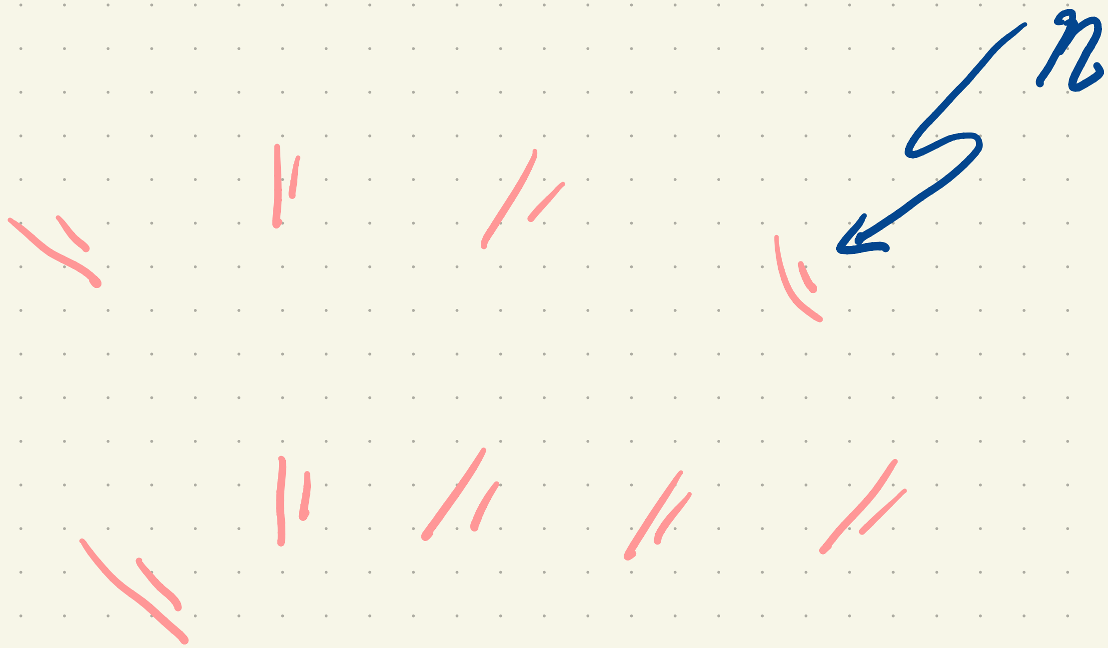
$$\int_\gamma dT = \int_a^b \frac{d}{dt}\, T(\gamma(t))\, dt$$

$$= T(\gamma(b)) - T(\gamma(a))$$

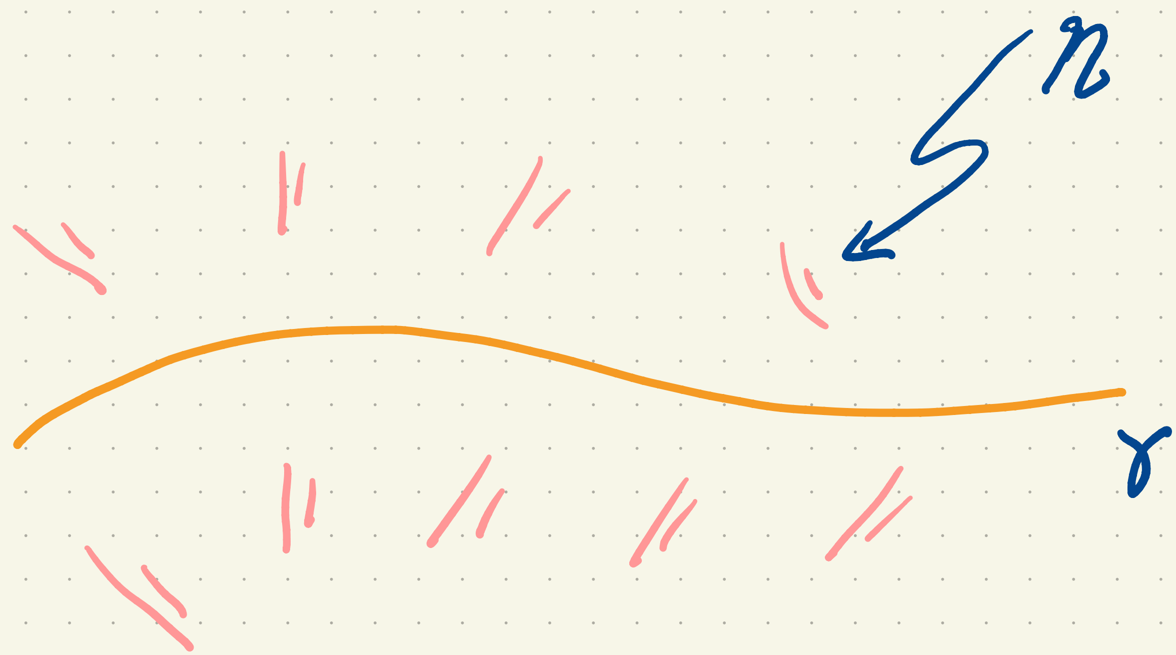
$$= T(p_1) - T(p_0)$$

FTC: $\int_a^b f'(s)\,ds = f(b) - f(a)$

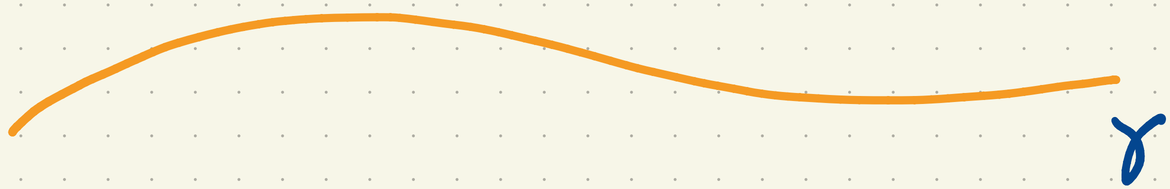
Derivative of a Covector Field



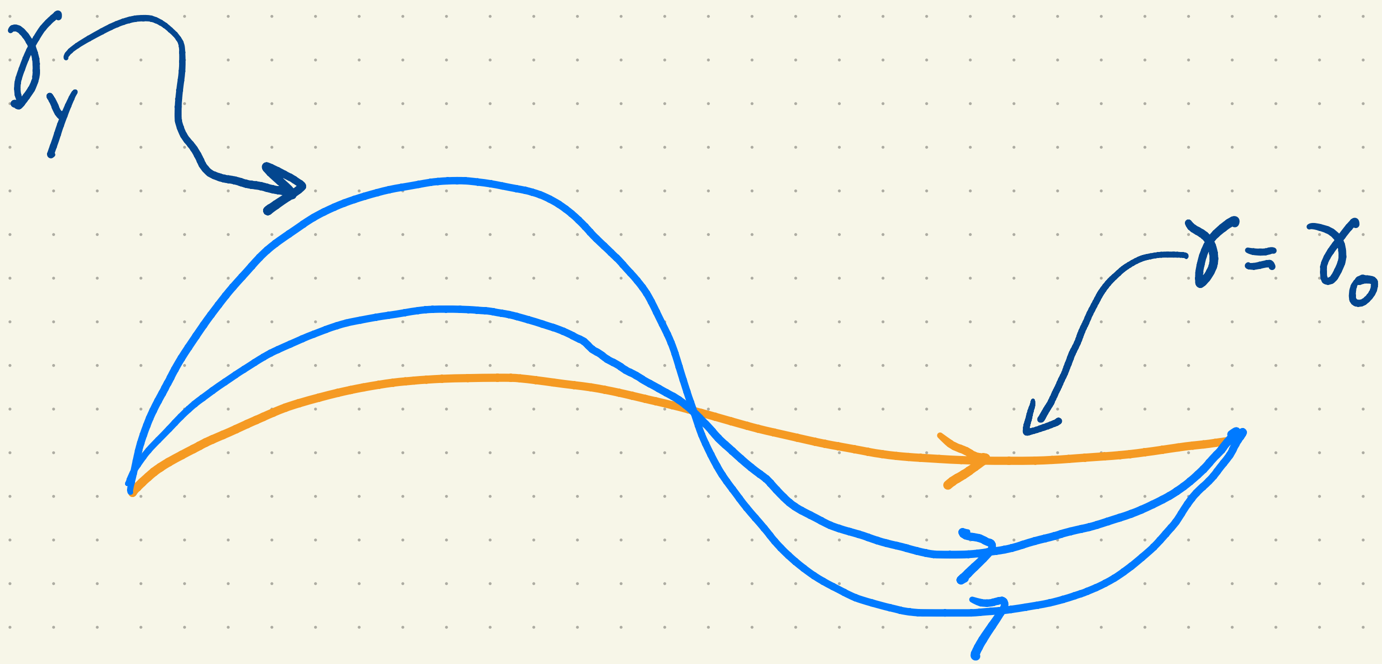
Derivative of a Covector Field



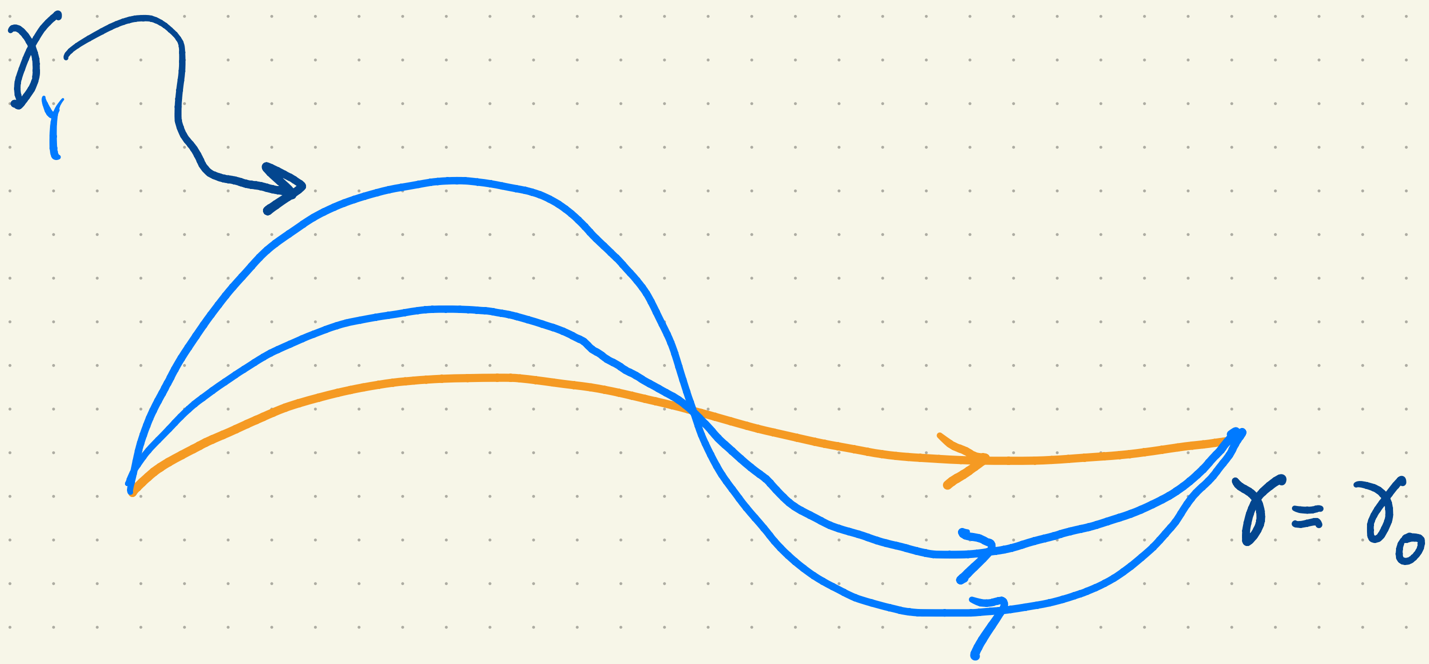
Derivative of a Covector Field



Derivative of a Covector Field

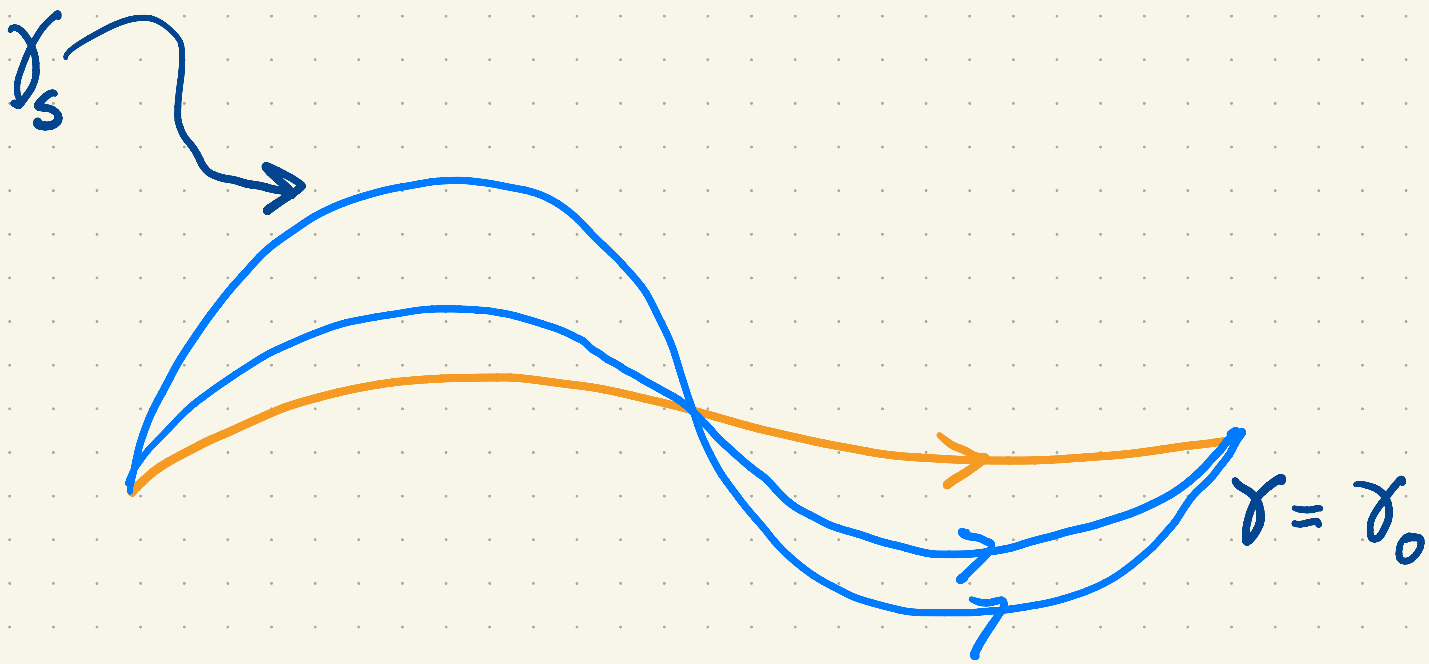


Derivative of a Covector Field



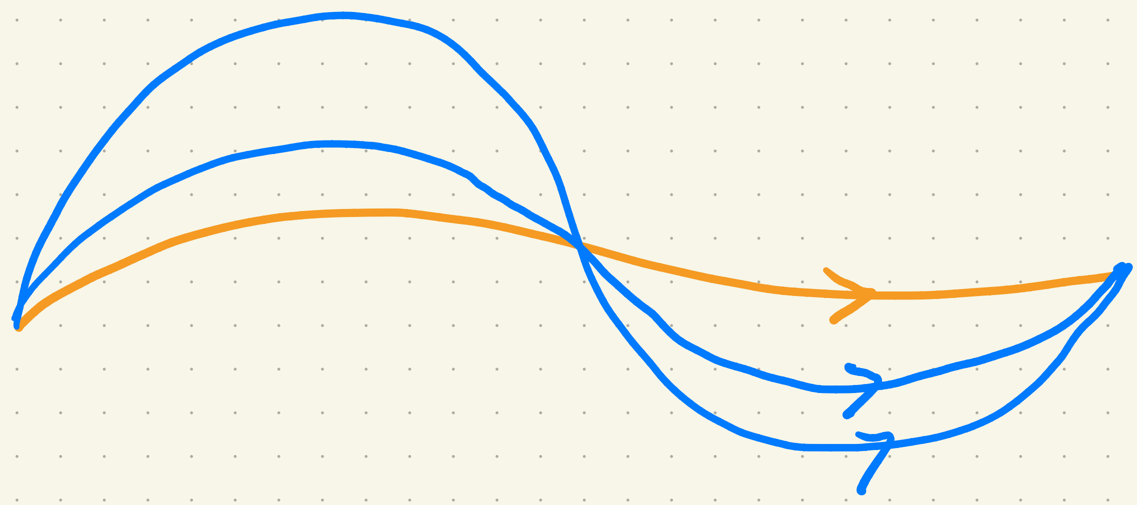
$$h(y) = \int_{\gamma_y} \eta$$

Derivative of a Covector Field



$$h(s) = \int_{\gamma_s} \eta \qquad \Bigg| \qquad h'(0) = \,?$$

Derivative of a Covector Field



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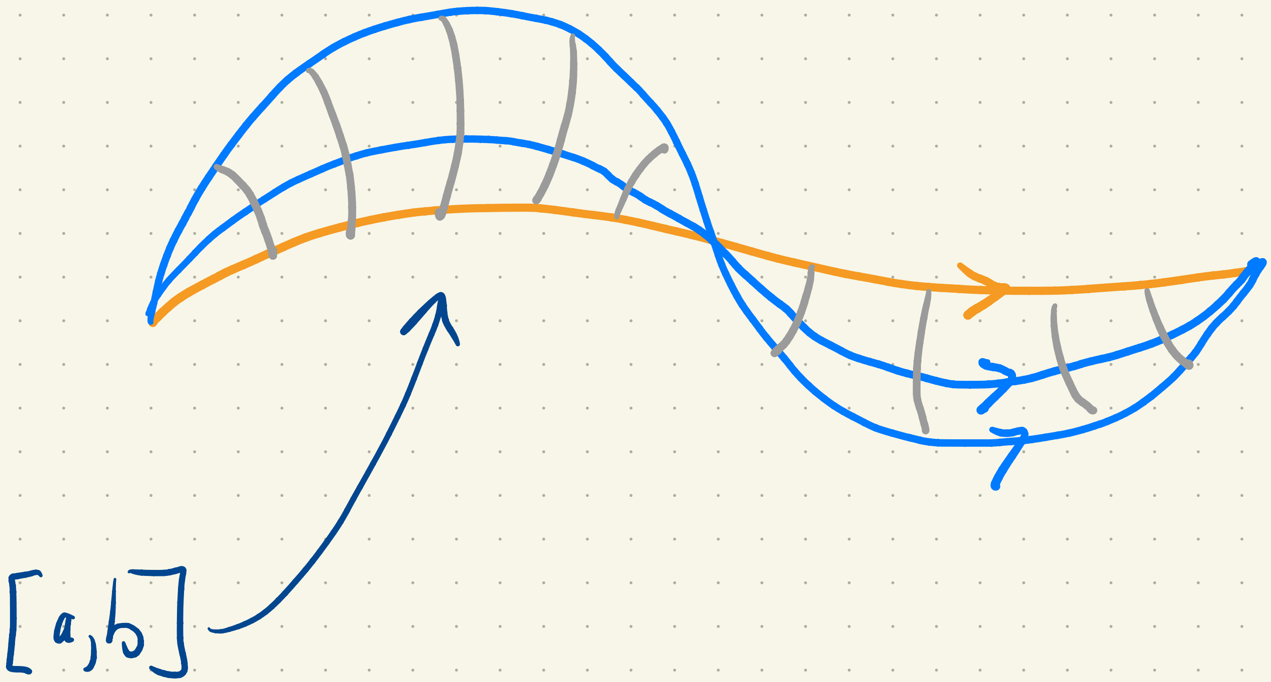

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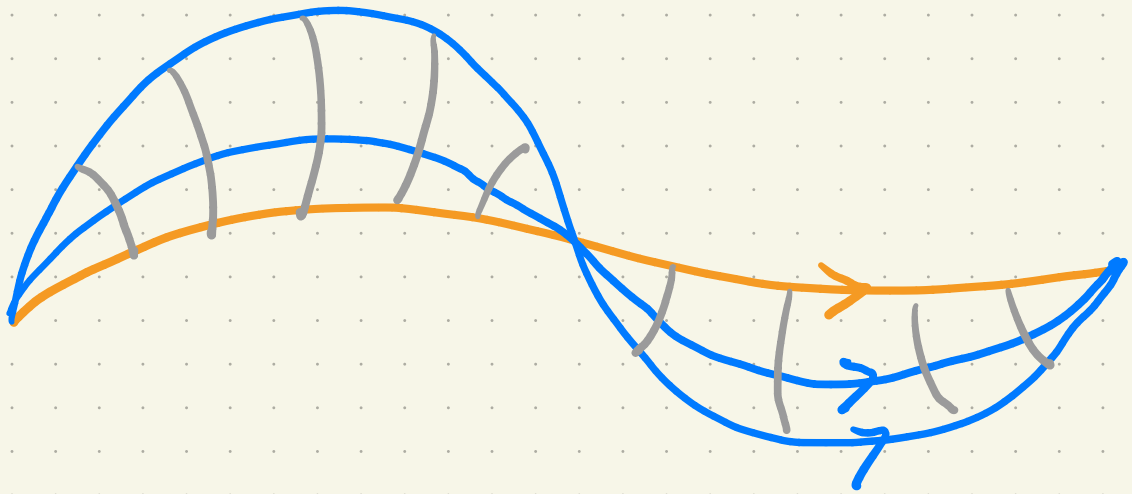

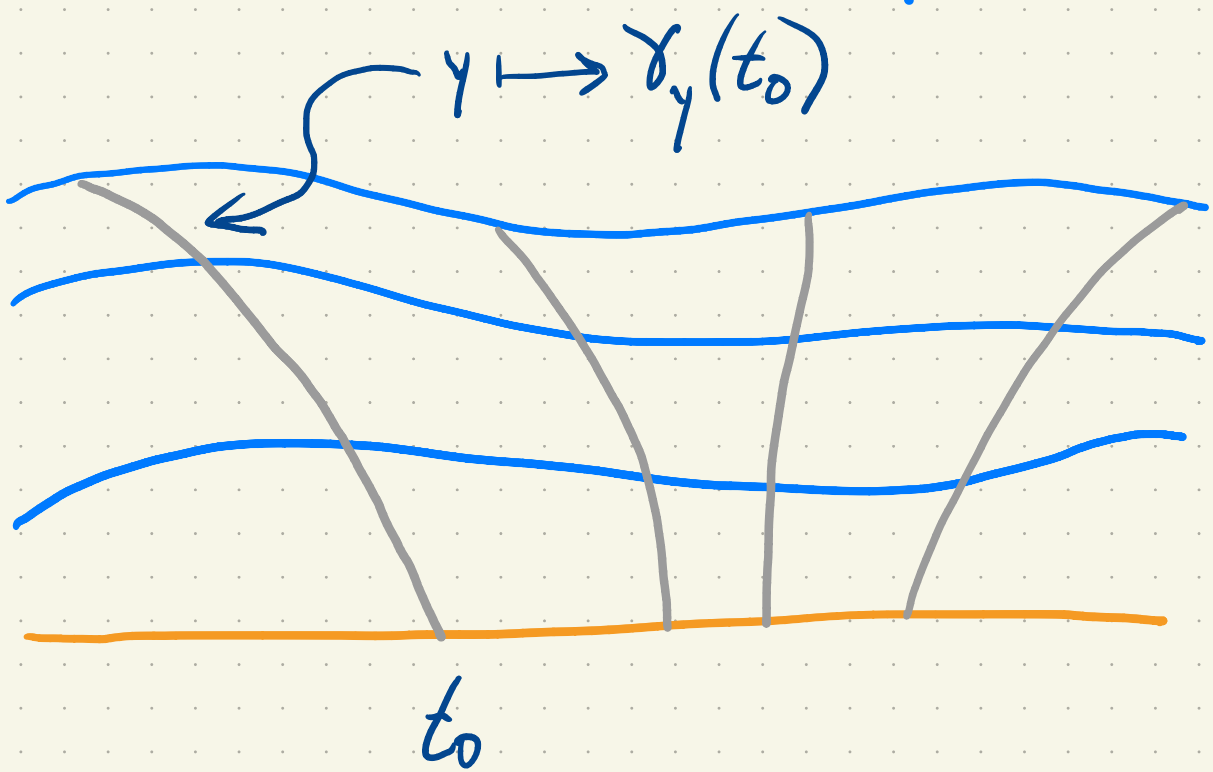

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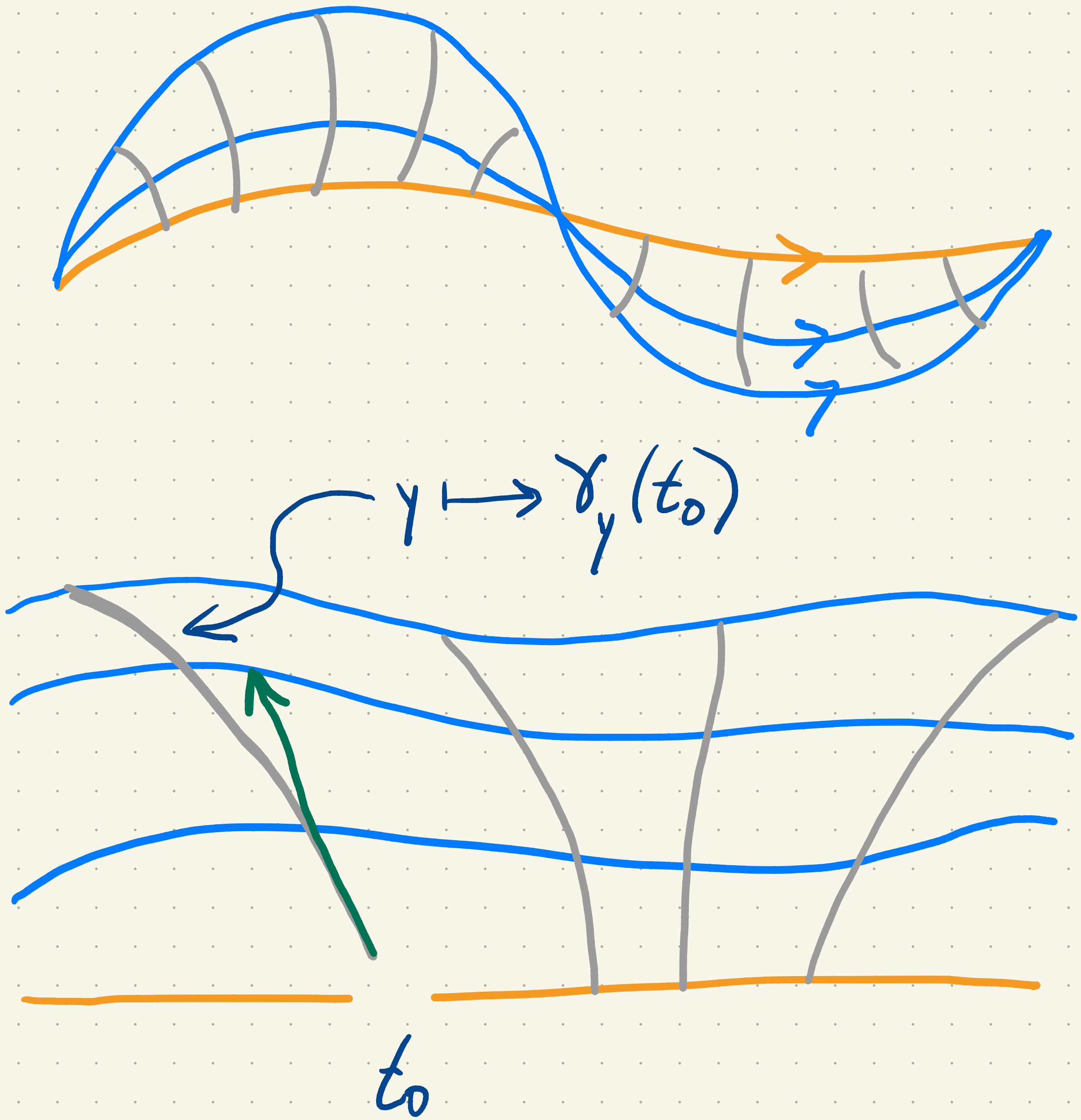

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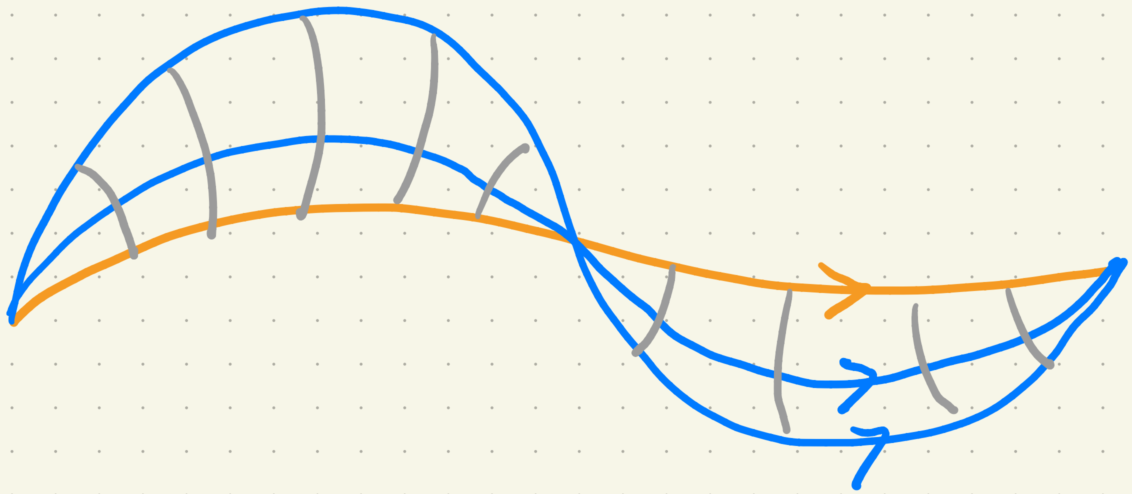
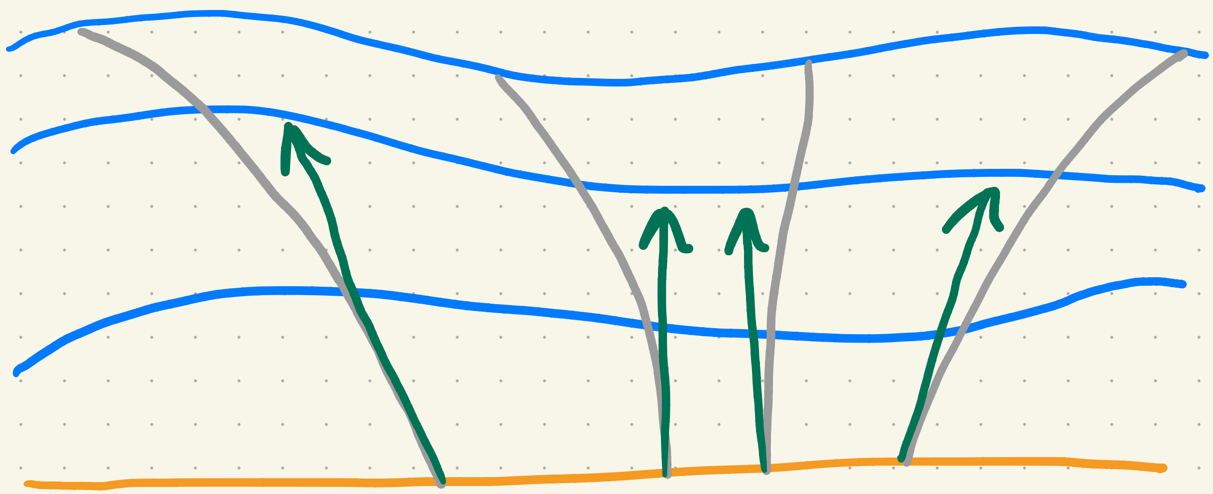
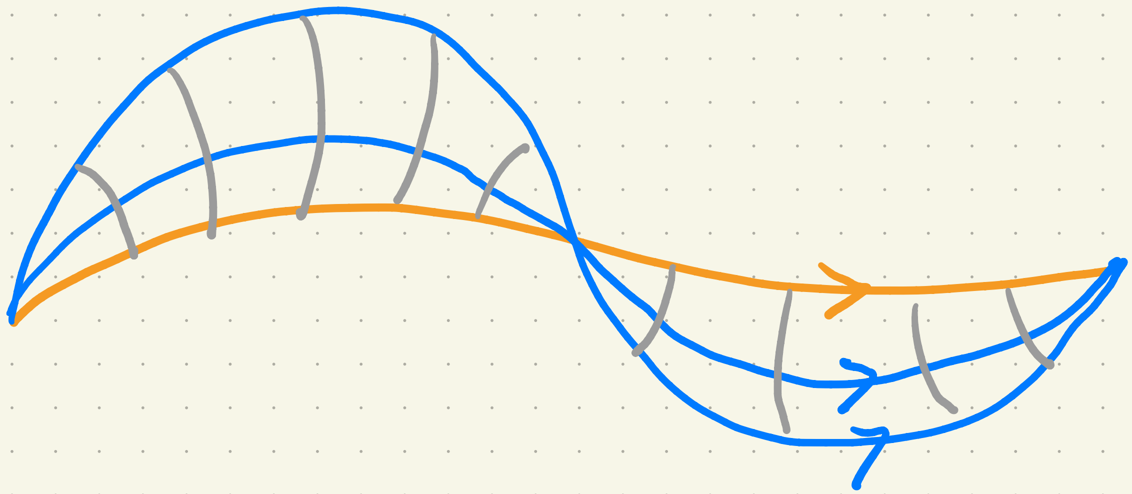
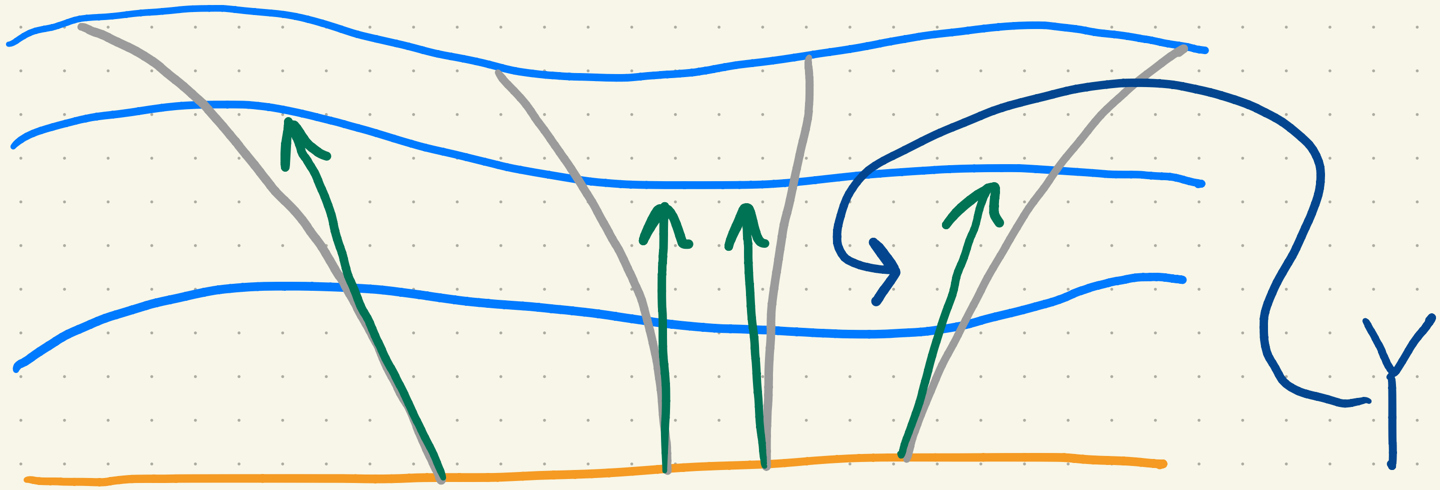




Derivative of a Covector Field





Exterior Derivative (Part 1)

- $\eta \xmapsto{d} d\eta$

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- $d\eta[Y, \cdot]$ is a covector

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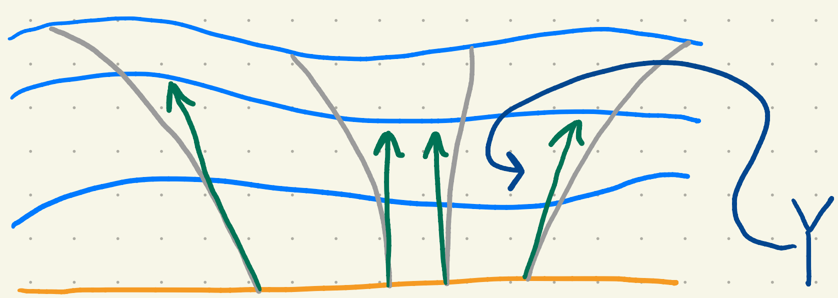
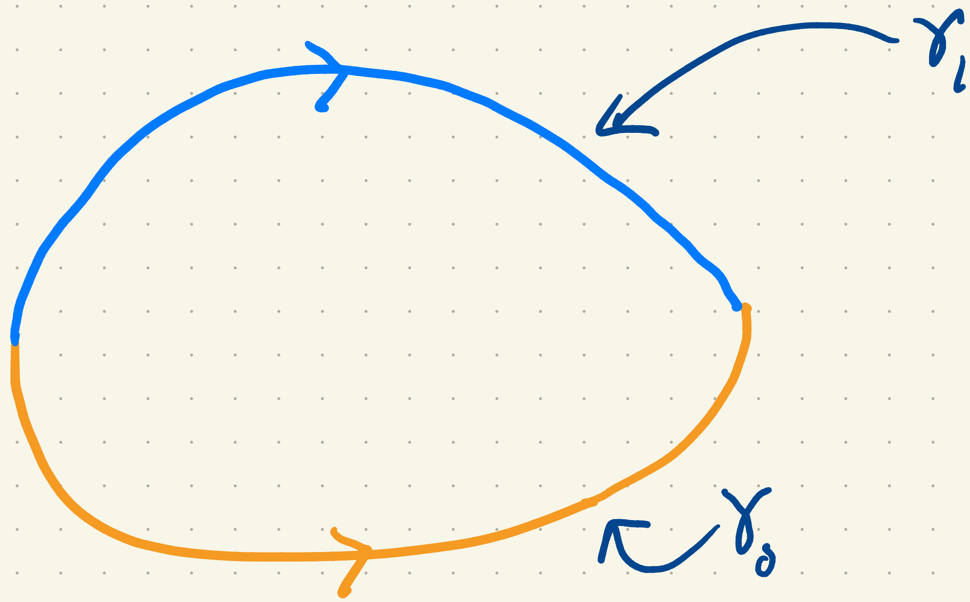
$$\int_\gamma \eta$$

- $\eta \xmapsto{d} d\eta$
- $d\eta[Y, \cdot]$ is a covector
- $\displaystyle\int_\gamma d\eta[Y, \cdot] = \frac{d}{dy}\Big|_{y=0} \int_{\gamma_y} \eta$

$h'(0)$

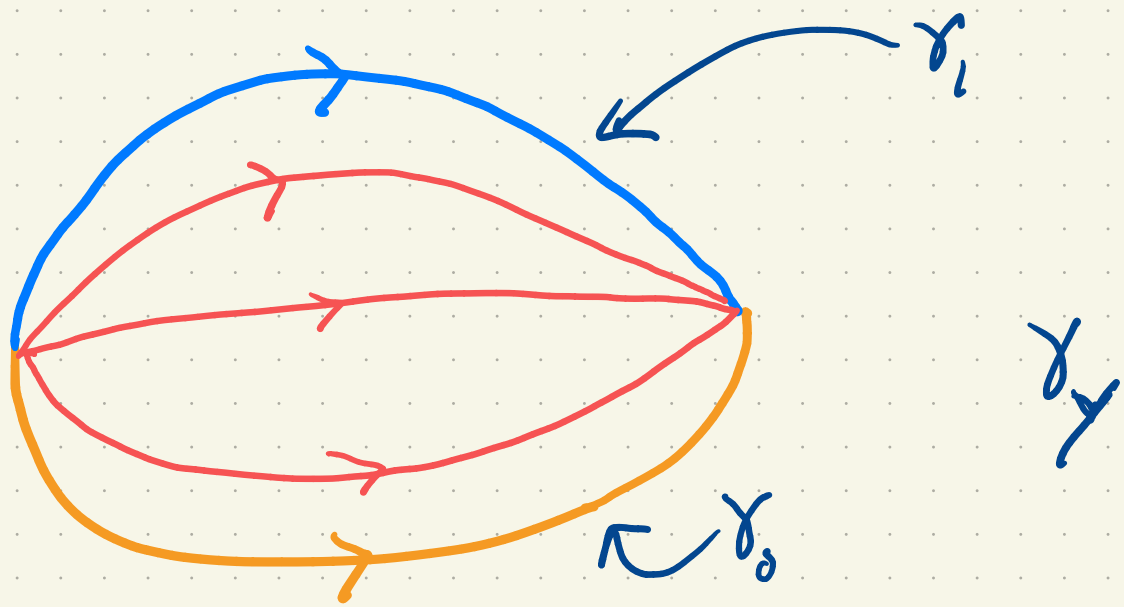


Stokes' Theorem (Preview)



$$\int_{\gamma_1} \eta - \int_{\gamma_0} \eta = \ ?$$

Stokes' Theorem (Preview)



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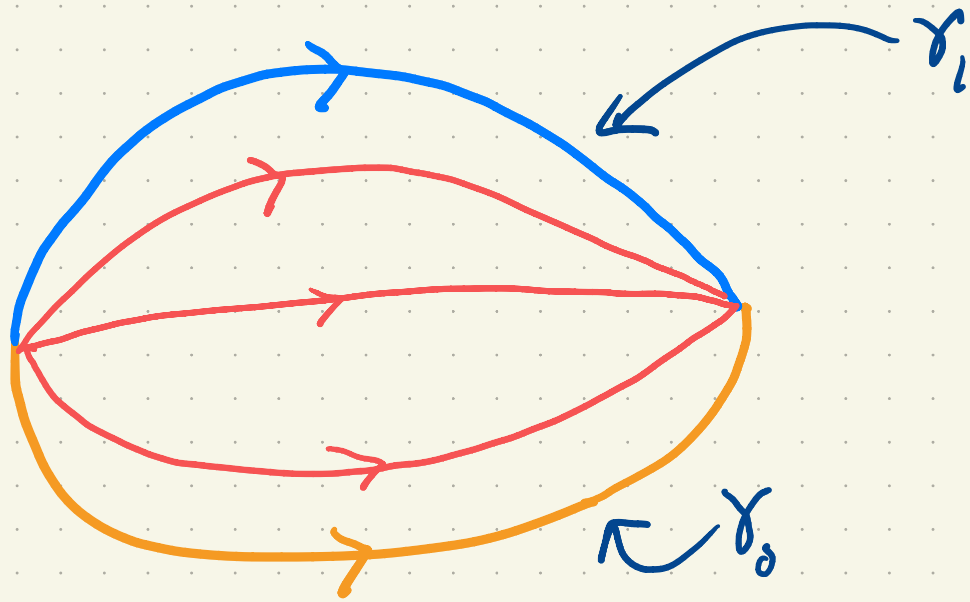
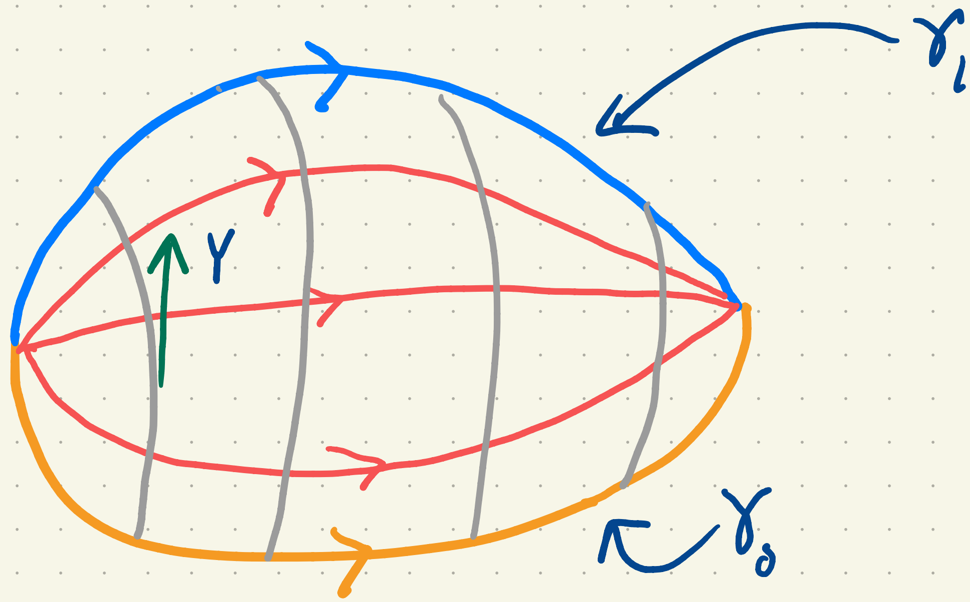
Stokes' Theorem (Preview)



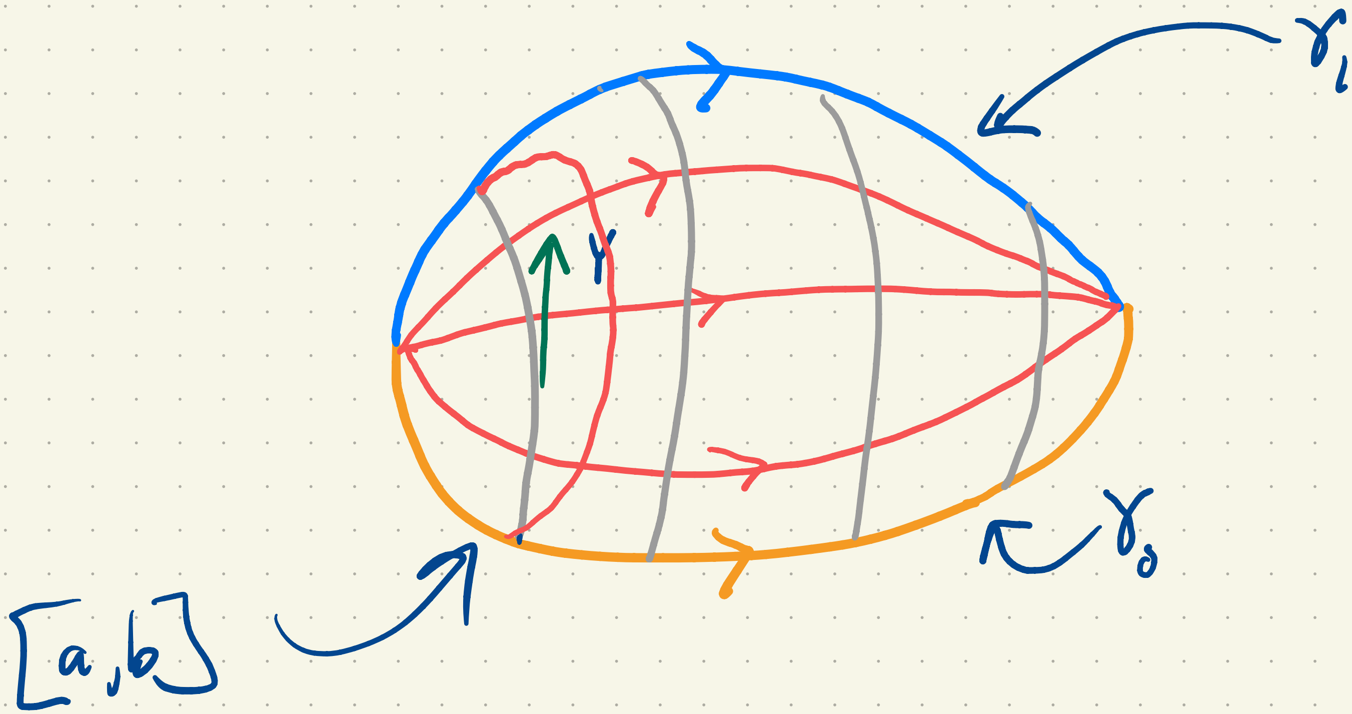
$$\int_{\gamma_1} \eta - \int_{\gamma_0} \eta = \int_0^1 \left[\frac{d}{dy} \int_{\gamma_y} \eta \right] dy$$

Stokes' Theorem (Preview)



$$\int_{\gamma_1} \eta - \int_{\gamma_0} \eta = \int_0^1 \left[\frac{d}{dy} \int_{\gamma_y} \eta \right] dy = \int_0^1 \int_{\gamma_y} d\eta(y, \cdot)\, dy$$

Stokes' Theorem (Preview)



$$\int_0^1\int_a^b d\eta(Y, \gamma_y')\,dt\,dy$$

$$\int_{\gamma_1}\eta - \int_{\gamma_0}\eta = \int_0^1\left[\frac{d}{dy}\int_{\gamma_y}\eta\right]dy = \int_0^1\int_a^b d\eta(Y,\gamma_y')\,dt\,dy$$