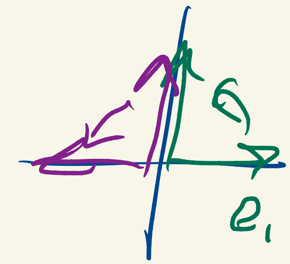
$$\mathbb{R}^2 \to \mathbb{R}^2$$

$$x \to Ax$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$e_1 \to e_2$$

$$e_2 \to -e_1$$



$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$$

$$= \underbrace{\lambda^2 + 1}_{\to\ \pm i}$$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

R_θ rotation by angle θ c.c.w.

$$e_1 \longrightarrow \cos\theta\, e_1 + \sin\theta\, e_2$$

$$e_2 \longrightarrow \cos\theta\, e_2 - \sin\theta\, e_1$$

$$r \geq 0$$

$$A = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$$

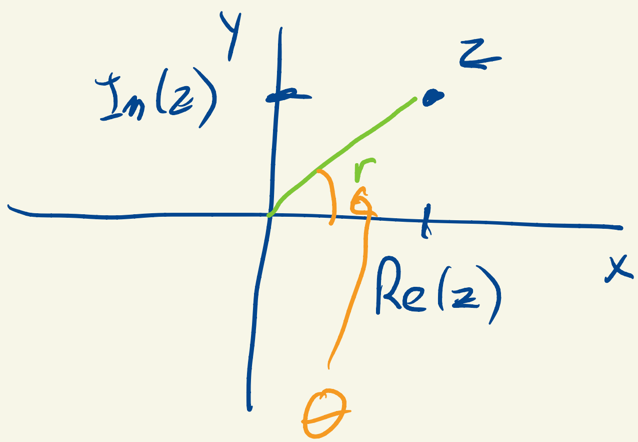
$$\boxed{\begin{aligned} e_1 &\longrightarrow r\cos\theta\, e_1 + r\sin\theta\, e_2 \\ e_2 &\longrightarrow r\cos\theta\, e_2 - r\sin\theta\, e_1 \end{aligned}}$$

Complex Numbers

x, y real

$$z = x + iy$$

$$i^2 = -1$$



$$Re(z) = x$$

$$Im(z) = y$$

$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \arctan(y/x) \ (\pm \pi)$$

$$z = r\cos\theta + i r \sin\theta$$

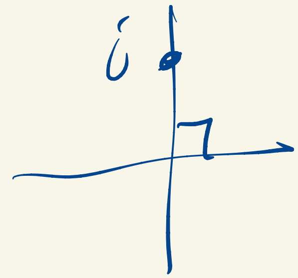
$$= r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$z_1 = r_1 e^{i\theta_1}$$

$$z_2 = r_2 e^{i\theta_2}$$

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2}$$

$$= r_1 r_2 e^{i(\theta_1+\theta_2)}$$



$$i \cdot i = -1$$

$$i = e^{i\pi/2} = \cos(\pi/2) + i \sin(\pi/2)$$

$$= 0 + i \cdot 1$$

$$= i$$

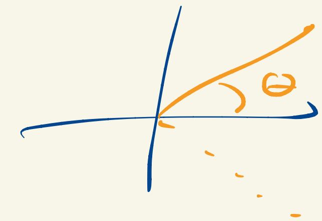
$$w = re^{i\theta}$$

$$z \longmapsto \underbrace{wz}_{\substack{z \text{ scaled by } r \\ \text{and rotated by } \theta}}$$

$$z = x + iy$$

$$i^2 = -1$$

$$(-i)^2 = -1$$



$$\bar{z} = x - iy$$

$$z\bar{z} = x^2 + y^2 = |z|^2$$

$$z = re^{i\theta}$$

$$\bar{z} = re^{-i\theta}$$

$$\overline{zw} = \bar{z}\,\bar{w}$$

$$A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} = 5 \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$$= 5 \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1$$

$$\theta = \arctan\left(4/3\right)$$

$$= 51.3^\circ$$

$$= 0.92\,\text{rad}$$

$$\det(A-\lambda I) = \det\begin{bmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 25$$

$$\lambda = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6 \pm \sqrt{-64}}{2}$$

$$= 3 \pm \frac{8i}{2}$$

$$= 3 \pm 4i$$

$$= 5e^{\pm i\theta} \qquad \theta = 0.9..$$

$$r = \sqrt{x^2+y^2}$$

$$\tan(\theta) = \frac{y}{x}$$

$$A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

$$\lambda = 3+4i$$

$$A - \lambda I = \begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix}$$

$$\begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$y = 1$

$$-4i\,x - 4 = 0$$

$$-4ix = 4$$

$$ix = -1$$

$$x = -1/i$$

$$= i$$

$\frac{1}{i} = -i$

$\frac{1}{7} \cdot 7 = 1$

$\frac{1}{22} \cdot 22 = 1$

$-i \cdot i = 1$

$1/i = -i$

$$v = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

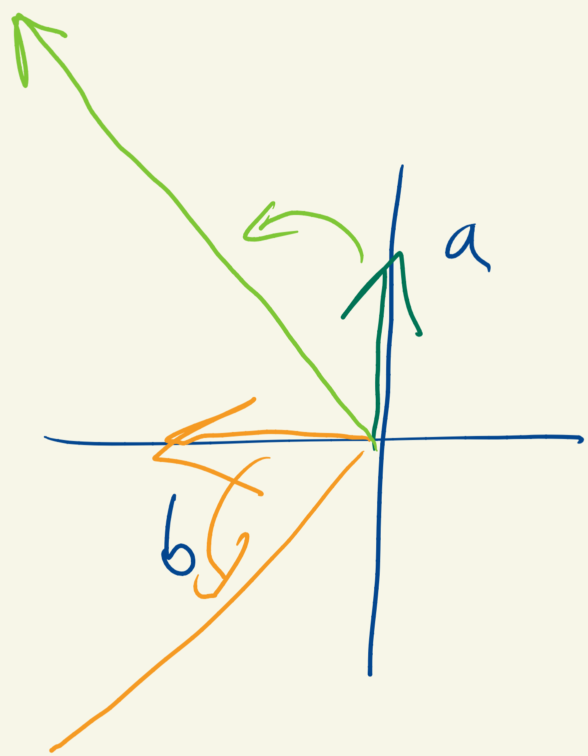
We look for complex eigenvectors

$$\lambda = 3+4i$$

$v = a - ib$ where a, b are real vectors

$$a = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad 3-4i = \bar{\lambda}$$

$$v = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$



$$\overline{Av} = \overline{\lambda v} = \bar{\lambda}\,\bar{v}$$

$$a = \frac{v+\bar{v}}{2} \qquad \frac{(x+iy)+(x-iy)}{2} = x$$

$$b = -\left(\frac{v-\bar{v}}{2i}\right) \qquad \frac{(x+iy)-(x-iy)}{2i} = y$$

$$Aa = A\left(\frac{v+\bar{v}}{2}\right) = \frac{Av + A\bar{v}}{2} = \frac{\lambda v + \bar{\lambda}\bar{v}}{2}$$

$$= \frac{\lambda v + \overline{\lambda v}}{2}$$

$$= \mathrm{Re}(\lambda v)$$

$$Ab = A\left(-\frac{v-\bar{v}}{2i}\right) = \frac{-Av + A\bar{v}}{2i}$$

$$= -\frac{\lambda v - \overline{\lambda v}}{2i}$$

$$= -\mathrm{Im}(\lambda v)$$

$$\lambda v = (r\cos\theta + ir\sin\theta)(a - ib)$$

$$= (r\cos\theta a + r\sin\theta b) + i(r\sin\theta a - r\cos\theta b)$$

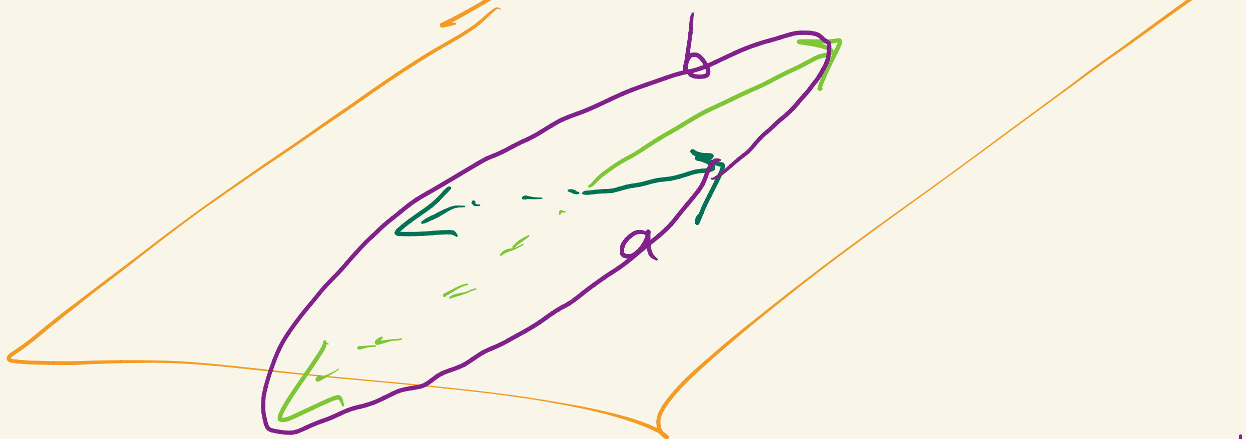
$$\mathrm{Re}(\lambda v) = r\cos\theta\, a + r\sin\theta\, b$$

$$\lambda = re^{i\theta}$$

$$-\mathrm{Im}(\lambda v) = r\cos\theta\, b - r\sin\theta\, a$$

$$a \longmapsto r\cos\theta\, a + r\sin\theta\, b$$

$$b \longmapsto r\cos\theta\, b - r\sin\theta\, a$$



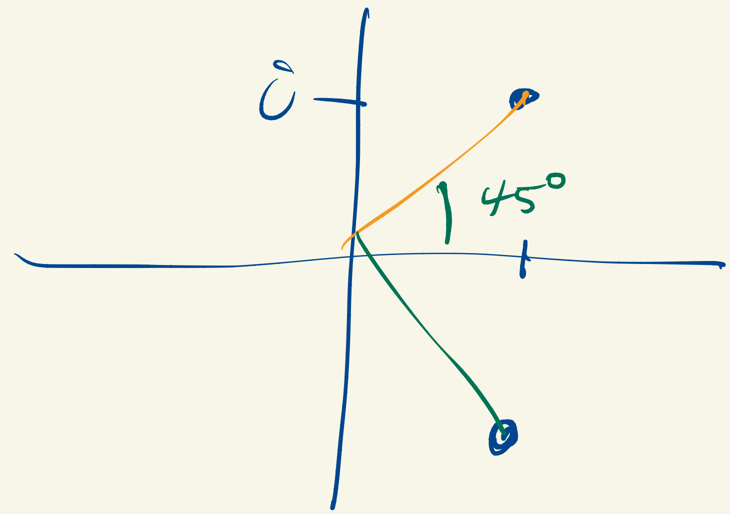
$$\lambda = r e^{i\theta}$$

$$v = a - ib$$

$$A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 2\lambda + 2$$

$$\lambda = 1 \pm i$$



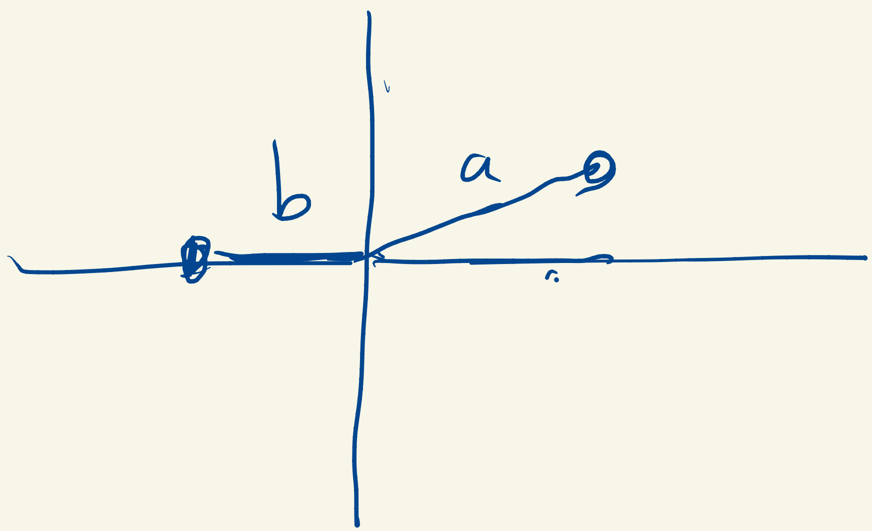
$$A - (1+i)I = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix}$$

$$x + (-2-i)y = 0$$

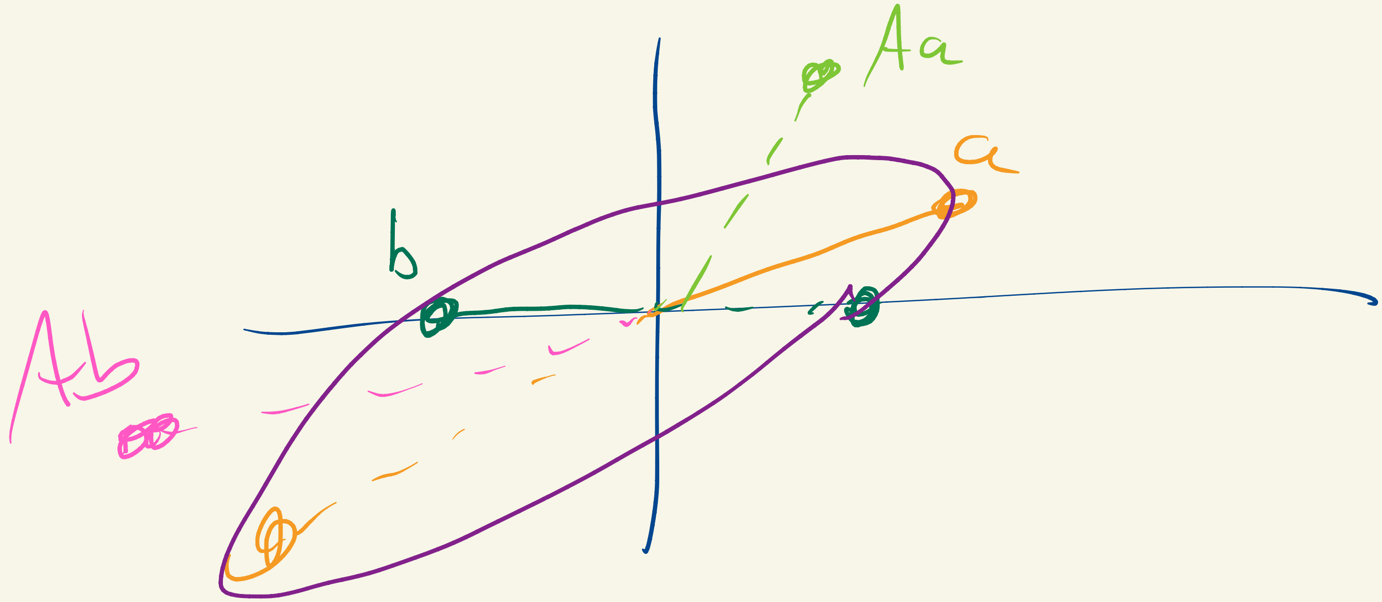
$$y = 1 \qquad x = 2+i$$

$$v = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} \qquad v = a - ib$$

$$a = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$



$$\lambda = \sqrt{2}\, e^{i\pi/4}$$



Aa
a
b
Ab