characteristic polynomial

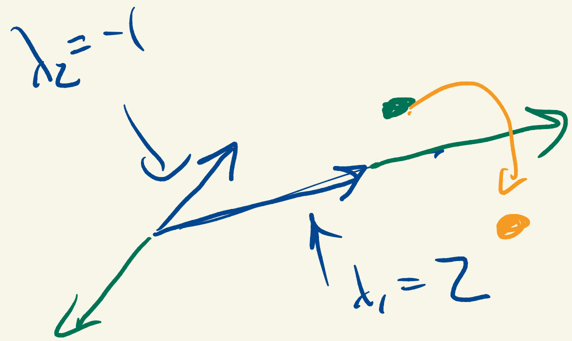
$$\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$$

set $\lambda = 0$ in

$$\boxed{\det(A) = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n}$$

$$Av = \lambda v$$

Hope: Find a basis of eigenvectors.



$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (3-\lambda)^2$$

eigenvalues: $\lambda = 3, 3$ $\qquad$ 0

$$A - 3I = \begin{array}{c} f \quad\;\; p \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

nullspace.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

all multiples of this.

When A has a repeated eigenvalue, A can fail to have a basis of eigenvectors.

$$A = \begin{bmatrix} 2 & 2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 0 \\ -2 & -3-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (-2-\lambda)\det\begin{pmatrix} 2-\lambda & 2 \\ -2 & -3-\lambda \end{pmatrix}$$

$$= (-2-\lambda)(\lambda+2)(\lambda-1)$$

$$= -(\lambda+2)^2(\lambda-1)$$

$$\lambda = 1, -2$$

$$A+2I = \begin{bmatrix} 4 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{array}{ccc} p & f & f \end{array} \begin{bmatrix} 4 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$$

$$\underbrace{c_1 v_1 + c_2 v_2}$$

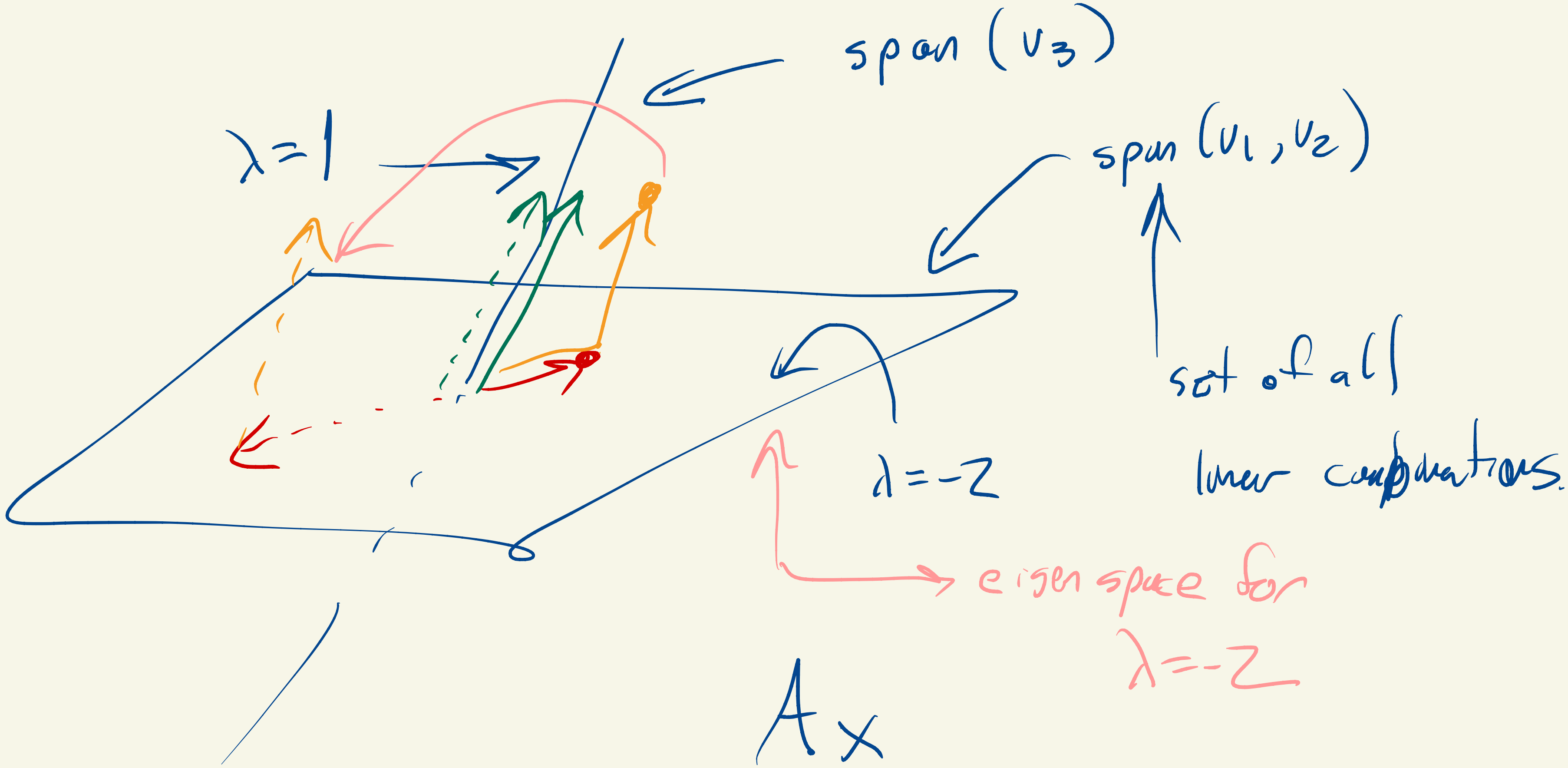
eigenvectors w/ eigenvalue -2

$$A - I = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\to \begin{array}{ccc} p & f & p \end{array}$$
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * \\ 1 \\ * \end{bmatrix} \qquad v_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$



Def: A ${}_{\wedge}$square matrix is diagonalizable if it admits a basis of eigenvectors

$A \quad n \times n$

$v_1, \ldots, v_n$ basis of eigenvectors

$\lambda_1, \ldots, \lambda_n$

$$A v_k = \lambda_k v_k$$

What happens to x? $\quad Ax = ?$

$$x = c_1 v_1 + \cdots + c_n v_n$$

$$Ax = A(c_1 v_1 + \cdots + c_n v_n)$$

$$= c_1 A v_1 + \cdots + c_n A v_n$$

$$= \lambda_1 c_1 v_1 + \lambda_2 c_2 v_2 + \cdots + \lambda_n c_n v_n$$

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\lambda_1 = 5 \qquad \lambda_2 = 1$$

$$Av = \lambda v$$

$$Av_1 = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \checkmark$$

$$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

$$AP = A\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} Av_1 & Av_2 \end{bmatrix}$$

$$= \begin{bmatrix} 5v_1 & 1v_2 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= P \underbrace{\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}}_{D}$$ eigenvalues

$$AP = PD$$

diagonal

$$P^{-1}AP = D \quad \text{or} \quad A = PDP^{-1}$$

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$$

$$P^{-1} \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = PDP^{-1}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\underbrace{P^{-1} A P} = D$$

job: e_k to v_k

$$e_1 \longrightarrow v_1 \longrightarrow \lambda_1 v_1 \longrightarrow \lambda_1 e_1$$

$$A = PDP^{-1}$$

$$Ax = (PD)\underbrace{P^{-1}x}$$

$$P^{-1}v_1 = e_1$$

$$P^{-1}v_2 = e_2$$

$$v_1 = Pe_1$$

$$v_2 = Pe_2$$

$$P^{-1}x = y \iff x = Py$$

$$x = P\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$x = y_1v_1 + y_2v_2 + \cdots + y_nv_n$$

$$Ax = \lambda x$$

$$\begin{aligned} A^3 x &= A^2(Ax) \\ &= A^2(\lambda x) \\ &= \lambda A^2 x \\ &= \lambda A(Ax) \\ &= \lambda A(\lambda x) \\ &= \lambda^2 Ax \\ &= \lambda^3 x \quad \checkmark \end{aligned}$$

$$A = P^{-1} D P$$

$$A^2 = P^{-1} D \underbrace{P\, P^{-1}}_{I} D P$$

$$= P^{-1} D^2 P$$

$$A^{100} = P^{-1} \underset{\substack{\uparrow \\ \text{easy to compute} \\ \text{easy to understand}}}{D^{100}} P$$

$A \quad 0<\lambda_k<1$

$A^{10}, A^{100}, A^{1000}, \ldots$

diagonal with λ's

$A = P^{-1} D P$

$A^{1000} = P^{-1} D^{1000} P$

$$\begin{bmatrix} \lambda_1^{1000} & & 0 \\ & \lambda_2^{1000} & \\ 0 & & \ddots \ \lambda_n^{1000} \end{bmatrix}$$

$A^N = \underbrace{P^{-1} D^N P}_{P^{-1}\, 0\, P \,=\, 0} \qquad N \to \infty$

$\lambda^2 + 1 = 0 \qquad \lambda = \pm i$