c) If X is an exchange matrix

$$\det(X) = -\det(I) = -1$$

d) If P is a permutation matrix

$$(d_1, 0, \ldots, 0)$$

$$\det(P) = \begin{cases} 1 \\ -1 \end{cases}$$

$$d_1 (1, 0, \ldots, 0)$$

$$(d_i\text{'s} \neq 0)$$

g) $$\det\begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix} = d_1 \det\begin{pmatrix} 1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$$

$$= d_1 \cdots d_n \det(I) = d_1 \cdots d_n$$

$$\det\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = ad$$

(case where $d_i = 0$ is easy: $\det(A) = 0$ since it has a zero row)

b) If A is upper triangular or lower triangular then $\det(A)$ is the product of the diagonal entries.

$$\det\begin{pmatrix} d_{11} & * & * & * \\ & d_{22} & & * \\ & & \ddots & \\ 0 & & & d_{nn} \end{pmatrix} = d_{11} d_{22} \cdots d_{nn}$$

$$\det\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = da - 0 \cdot b = ad$$

$d_{ii} \neq 0$

$$\begin{pmatrix} d_{11} & \square & * & \cdots & * \\ 0 & d_{22} & * & \cdots & * \\ & & d_{33} & & \\ & 0 & & \ddots & \\ & & & & d_{nn} \end{pmatrix} \longrightarrow \begin{pmatrix} d_{11} & 0 & *' & \cdots & *' \\ 0 & d_{22} & * & \cdots & * \\ & & d_{33} & & \\ & 0 & & \ddots & \\ & & & & d_{nn} \end{pmatrix}$$

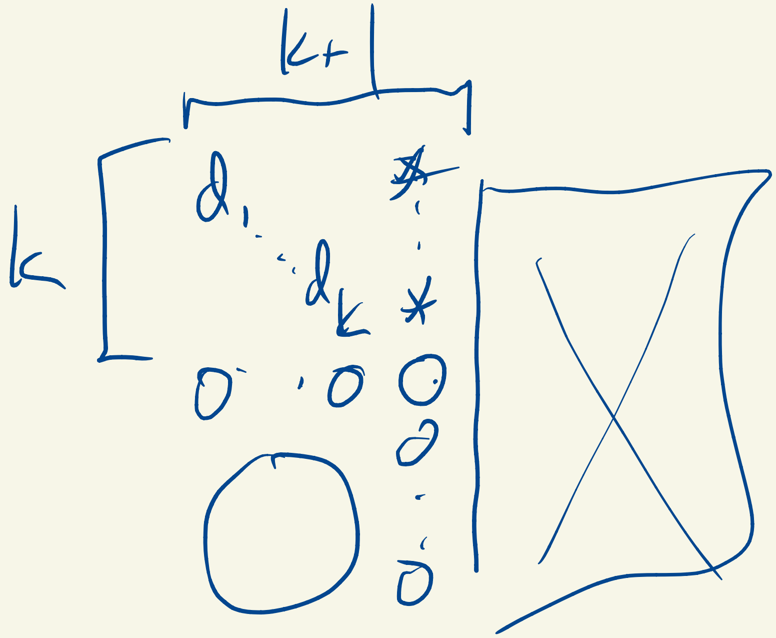
det( )

det does not change

$=$

det( )

$$\begin{pmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{nn} \end{pmatrix}$$



If some diagonal entry is 0, the cols are linearly dep and $\det(A) = 0$

product of diagonal entries

i) A, B

$$\det(AB) = \det(A)\det(B)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$(ad-bc) \qquad (eh-fg)$$

$$(ae+bg)(cf+dh) - (af+bh)(ce+dg)$$

$$\cancel{aecf} + aedh + bgcf + \cancel{bgdh}$$
$$- \cancel{afce} - afdg - bhce - \cancel{bhdg}$$

$$(ad-bc)(eh-fg) = adeh + bcfg - bceh - adfg$$

$\det(B) \neq 0$

$$\det{}'(A) = \frac{\det(AB)}{\det(B)}$$

1) $\det'(I) = 1$
2) $\det'(XA) = -\det'(A)$
3) $\det'$ is linear in each row separately

$$\det{}' = \det$$

$$\det(A) = \frac{\det(AB)}{\det(B)}$$

$$\det(AB) = \det(A)\det(B)$$

$$\det{}'(I) = \frac{\det(IB)}{\det(B)} = \frac{\det(B)}{\det(B)} = 1 \quad !$$

$$\det{}'(XA) = \frac{\det(XAB)}{\det(B)} = \frac{-\det(AB)}{\det(B)} = -\det{}'(A)$$

$$\det{}'(v_1+v_1', v_2, \ldots, v_n) = \det{}'\left(\begin{bmatrix} (v_1+v_1')^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} \right)$$

$$\begin{bmatrix} (v_1+v_1')^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} B = \begin{bmatrix} (v_1+v_1')^T B \\ v_2^T B \\ \vdots \\ v_n^T B \end{bmatrix}$$

$$\frac{\det(AB)}{\det(B)} = \frac{\det\begin{pmatrix} v_1^T B \\ \vdots \\ v_n^T B \end{pmatrix}}{\det(B)} + \frac{\det\begin{pmatrix} (v_1')^T B \\ \vdots \\ v_n^T B \end{pmatrix}}{\det(B)}$$

$$\det{}'(v_1+v_1', v_2, \ldots, v_n) = \det{}'(v_1, v_2, \ldots, v_n) + \det{}'(v_1', v_2, \ldots, v_n)$$

$$\det(A A^{-1}) = \det(I) = 1$$

$$\hookrightarrow \det(A) \cdot \det(A^{-1})$$

$$\det(A^{-1}) = 1/\det(A)$$

If A has an inverse then $\det(A) \neq 0$.

$\det(A) = 0 \iff A$ does not have an inverse

$\det(A) \neq 0 \iff A$ has an inverse.

j) Suppose $A = L\,U$

lower tri 1's on diag

upper triangular

$$\det(A) = \det(L)\det(U)$$

$$= 1 \cdot \text{product of diagonal entries of } U$$

$$PA = LU$$

$$\underbrace{\det(P)}_{\pm 1} \det A = \text{prod of diag entries of } U$$

k) $A = \begin{bmatrix} 1 & 0 \\ 0 & A' \end{bmatrix}$

$$\det(A) = \det(A')$$

$$A' = LU$$

$$\begin{bmatrix} 1 & 0 \\ 0 & A' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & LU \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix}$$

$$\det(A) = \det\begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix} \cdot \det\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$$

lower tri, 1's on diag

upper tri

$$= 1 \cdot (1 \cdot \text{diag entries of } U)$$

$$= \text{product of diag entries of } U$$

$$= \det(A')$$

$$\det\begin{pmatrix} \boxed{2} & \boxed{3} & \boxed{7} \\ * & * & * \\ * & * & * \end{pmatrix} = \det\begin{pmatrix} 2 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} + \det\begin{pmatrix} 0 & 3 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & 7 \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$= \det\begin{pmatrix} 2 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} + \begin{pmatrix} 0 & 3 & 0 \\ * & 0 & * \\ * & 0 & * \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 7 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}$$

$$= 2 \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} + \text{---} + \text{---}$$

$$\det \begin{pmatrix} 0 & 3 & 0 \\ * & 0 & x \\ * & 0 & x \end{pmatrix} = -\det \begin{pmatrix} 3 & 0 & 0 \\ 0 & * & x \\ 0 & * & y \end{pmatrix}$$

$$= -3 \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$