So, if A^TA has an inverse,

then the columns of A are linearly independent.

$$Ax = 0$$

$$x = 0$$

A, columns are linearly independent

$$A^+ = (A^TA)^{-1}A^T$$

Moore-Penrose inverse

Pseudo inverse

Claim: A^+ is a left inverse of A.

$$A^+A = \left((A^TA)^{-1}A^T\right)A$$

$$= (A^TA)^{-1}(A^TA)$$

$$= I$$

What is A^+ if A is square?

A^+ is a left inverse of A.

So A^+ had better be A^{-1}!

$$A^+ = (A^TA)^{-1}A^T$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$= A^{-T}$$

$$(A^TA)^{-1}A^T = A^{-1}(A^T)^{-1}A^T$$

$$= A^{-1}I$$

$$= A^{-1}$$

If

$$Ax = b$$

$$\underbrace{A^{+}A}\,x = A^{+}b$$

$$Ix = A^{+}b$$

$$x = A^{+}b$$

$$A^{+} = (A^{T}A)^{-1}A^{T}$$

We can compute $A^{+}b$ using the QR factorization of A

$$A = QR$$

$$A^{T} = (QR)^{T}$$

$$= R^{T}Q^{T}$$

$$A^{T}A = R^{T}Q^{T}QR$$

$$= R^{T}R$$

$$(A^{T}A)^{-1} = (R^{T}R)^{-1}$$

$$= R^{-1}(R^T)^{-1}$$

$$A^+ = (A^TA)^{-1}A^T = R^{-1}(R^T)^{-1}(QR)^T$$

$$= R^{-1}(R^T)^{-1}R^TQ^T$$

$$= R^{-1}IQ^T$$

$$= R^{-1}Q^T$$

A^+b

$$\boxed{A^+ = R^{-1}Q^T}$$

$z = A^{+}b$ means $z = R^{-1}Q^{T}b$

$$Rz = \underbrace{RR^{-1}}_{I}Q^{T}b$$

$$Rz = Q^{T}b$$

$$z = A^{+}b \iff Rz = Q^{T}b$$

So: to compute $A^{+}b$

1) form $Q^{T}b$

2) solve $Rz = Q^{T}b$ using back substitution

Then $z = A^{+} b.$

If A is square and I give you $A = QR$

and want you to solve $Ax = b$ for x

what do you do?

$$QRx = b$$

$$\underbrace{Q^{T}Q}_{I}Rx = Q^{T}b$$

$$Rx = Q^{T}b$$

Now solve for x by back substitution.

A $\quad m \times n$

$Q^T Q = I$

cols of Q are orthonormal

$A = [a_1 \cdots a_n]$

$a_1 \cdots a_n \longrightarrow q_1 \cdots q_n$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \end{array} \; \begin{array}{cc} n & \begin{bmatrix} r_{11} & & * \\ & \ddots & \\ 0 & & r_{nn} \end{bmatrix} \end{array}$$

$$a_1 \qquad \tilde{q}_1 = a_1 \qquad q_1 = \tilde{q}_1 / \|\tilde{q}_1\|$$

$$\tilde{q}_2 = a_2 - (q_1^T a_2) q_1 \qquad a_2 = \tilde{q}_2 / \|\tilde{q}_2\|$$

$$\tilde{q}_2 = \|\tilde{q}_2\| \cdot q_2$$

$$a_2 = \|\tilde{q}_2\| q_2 + (q_1^T a_2) q_1$$

$$R = \begin{bmatrix} \|\tilde{q}_1\| & (q_1^T a_2) & (q_1^T a_3) \\ 0 & \|\tilde{q}_2\| & (q_2^T a_3) \\ \vdots & 0 & \|\tilde{q}_3\| \\ 0 & \vdots & \end{bmatrix}$$

What about wide matrices?

$$\begin{array}{cc} & n \\ m & \left[\begin{array}{ccc} \quad & \quad & \quad \end{array} \right] \end{array} \qquad m < n$$

right inverse.

We will build a left inverse for A^T

We'll assume that A has linearly independent rows.

Then A^T has linearly independent columns.

So $(A^T)^T A^T = AA^T$ is invertible.

Now we define

$$A^+ = A^T(AA^T)^{-1}$$

$$AA^+ = AA^T(AA^T)^{-1} = I$$

$Ax = b$ $\quad\quad\quad\quad$ $x = A^+ b$

Then $Ax = AA^+b = Ib = b.$

$\left[\qquad \right] \leftarrow$ more than one solution.

Next class: How to compute A^+ using QR factorization for wide matrices.