


So, if  $A^T A$  has an inverse,

then the columns of  $A$  are linearly independent.

$$Ax = 0$$
$$x = 0$$


$A$ , columns are linearly independent

$$A^+ = (A^T A)^{-1} A^T$$

Moore-Penrose inverse

Pseudo inverse

Claim:  $A^+$  is a left inverse of  $A$ .

$$\begin{aligned}
 A^+ A &= ((A^T A)^{-1} A^T) A \\
 &= (A^T A)^{-1} (A^T A) \\
 &= I
 \end{aligned}$$

$$A^+ = (A^T A)^{-1} A^T$$

$$\begin{aligned}
 (A^T)^{-1} &= (A^{-1})^T \\
 &= A^{-T} \quad \uparrow
 \end{aligned}$$

What is  $A^+$  if  $A$  is square?

$A^+$  is a left inverse of  $A$ .

So  $A^+$  had better be  $A^{-1}$ !

$$\begin{aligned}
 (A^T A)^{-1} A^T &= A^{-1} (A^T)^{-1} A^T \\
 &= A^{-1} I \\
 &= A^{-1}
 \end{aligned}$$

If

$$Ax = b$$

$$\underbrace{A^T A} x = A^T b$$

$$I x = A^T b$$

$$x = A^T b$$

$$A^+ = (A^T A)^{-1} A^T$$

We can compute  $A^+ b$   
using the QR factorization  
of  $A$

$$A = QR$$

$$\begin{aligned} A^T &= (QR)^T \\ &= R^T Q^T \end{aligned}$$

$$\begin{aligned} A^T A &= R^T Q^T Q R \\ &= R^T R \end{aligned}$$

$$(A^T A)^{-1} = (R^T R)^{-1}$$

$$= R^{-1} (R^T)^{-1}$$

$$A^{\dagger} = (A^T A)^{-1} A^T = R^{-1} (R^T)^{-1} (QR)^T$$

$$= R^{-1} (R^T)^{-1} R^T Q^T$$

$$= R^{-1} I Q^T$$

$$= R^{-1} Q^T$$

$A^{\dagger} b$

$$A^{\dagger} = R^{-1} Q^T$$

$$z = A^+ b \quad \text{means} \quad z = R^{-1} Q^T b$$

$$Rz = \underbrace{RR^{-1}}_I Q^T b$$

$$Rz = Q^T b$$

$$z = A^+ b \quad \iff \quad Rz = Q^T b$$

So: to compute  $A^+ b$

- 1) form  $Q^T b$
- 2) solve  $Rz = Q^T b$   
using back substitution

$$\text{Then } z = A^+ b.$$

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If  $A$  is square and I give you  $A = QR$   
and want you to solve  $Ax = b$  for  $x$

What do you do?

$$QRx = b$$

$$\underbrace{Q^T Q}_{I} R x = Q^T b$$

$$R x = Q^T b$$

Now solve for  $x$  by back substitution,

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$$A \quad m \times n$$

$$A = [a_1 \dots a_n]$$

$$Q^T Q = I$$

↑ cols of  $Q$   
are orthonormal

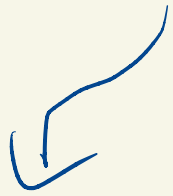
$$a_1 \dots a_n \longrightarrow q_1 \dots q_n$$

$$\xleftrightarrow{n}$$

$$m \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix} \xrightarrow{n} \begin{bmatrix} r_{11} & * & \\ & \ddots & \\ 0 & & r_{nn} \end{bmatrix}$$

$$a_1 \quad \tilde{q}_1 = a_1 \quad q_1 = \tilde{z}_1 / \|\tilde{z}_1\|$$

$$\tilde{q}_2 = a_2 - (q_1^T a_2) q_1 \quad a_2 = \tilde{z}_2 / \|\tilde{z}_2\|$$



$$\tilde{q}_2 = \|\tilde{z}_2\| \cdot q_2$$

$$a_2 = \|\tilde{q}_2\| q_2 + (q_1^T a_2) q_1$$

$$R = \begin{bmatrix} \|\tilde{q}_1\| & (q_1^T a_2) & (q_1^T a_3) \\ 0 & \|\tilde{q}_2\| & (q_2^T a_3) \\ \vdots & 0 & \vdots \\ 0 & \vdots & \|\tilde{q}_3\| \end{bmatrix}$$



What about wide matrices?

$$m \begin{matrix} n \\ \left[ \right] \end{matrix} \leftarrow \begin{matrix} m < n \\ \text{right inverse.} \end{matrix}$$

We will build a left inverse for  $A^T$

We'll assume that  $A$  has linearly independent rows.

Then  $A^T$  has linearly independent columns.

So  $(A^T)^T A^T = A A^T$  is invertible.

Now we define

$$A^+ = A^T (A A^T)^{-1}$$

$$A A^+ = A A^T (A A^T)^{-1} = \underline{I}$$

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$$A x = b$$

$$x = A^+ b$$

$$\text{Then } A x = A A^+ b = I b = b.$$

$[ \quad ] \leftarrow$  more than one solution.

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Next class: How to compute  $A^+$  using QR factorization for wide matrices.