Last class: $Ax = b$ Inverses

A

1) Left inverse X if

$$XA = I$$

2) Right inverse Y if

$$AY = I$$

3) Two sided inverse (inverse) Z

$$ZA = I$$
$$AZ = I$$

1) $\boxed{\text{If } A \text{ has a left inverse then the columns of } A \text{ are linearly independent}}$

(In fact the converse is also true! Stay tuned!)

Consequence: only square or tall matrices can have left inverses

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} | & | & | & | & | & | & | \end{bmatrix} \end{array}$$

$\mathbb{R}^m$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0$$

$$A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$$

$$Ax = 0$$

show the only solution is $x = 0$.

Suppose $Ax = 0$

Then $\underbrace{XA}_{I}x = X0$

$Ix = 0$

$x = 0$

The only solution of $Ax = 0$ is $x = 0$.

(The columns of A are linearly independent)

2) X is a left inverse of A if and only if X^T is a right inverse of A^T.

Suppose X is a left inverse of A

$$A^T X^T = (XA)^T = I^T = I$$

If X is a left inverse of A then

X^T is a right inverse of A^T.

3) 1) + 2) If A has a right inverse

then its rows are linearly independent

(If A has a right inverse then A^T has a left inverse

so the columns of A^T are linearly independent.

But the columns of A^T are the rows of A)

Only square and wide matrices can have

right inverses.

4) Only square matrices can have inverses.

5) If a square matrix A has a left inverse X and a right inverse Y then $X = Y$

Observe $XAY = X(AY) = XI = X$

$$XAY = (XA)Y = IY = Y$$

$\Rightarrow X = Y$

6) A square matrix has at most one inverse

Suppose W and Z are two inverses of A

Then W is a left inverse of A and

Z is a right inverse of A.

So by the previous property $W = Z$.

Notation: for a square matrix A, A^{-1} is it's one and only inverse (if it exists)

7) If A has a left inverse X then

$Ax = b$ has **at most one solution**

which, if it exists, is Xb.

Suppose $Ax = b$.

Then $XAx = Xb$ so $Ix = Xb$

so $x = Xb$.

If $Ax = b$ then $x = Xb$.

8) If A has a right inverse Y then

$Ax = b$ has $\boxed{\text{at least one solution.}}$

Given b let $x = Yb$. Then $Ax = AYb$

$= Ib$

$= b$

We just showed $Ax = b$ if $x = Yb$.

9) If A has an inverse (A is square!)

$$Ax = b$$

There exists a solution and indeed only one solution,
($Ax=b$ has exactly one solution)

10) Some square matrices don't have inverses.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$\det(A) = 1\cdot 4 - 2\cdot 2 = 0$
↳ we will see for a 2×2, there is an inverse $\Leftrightarrow \det A \neq 0$,

If a matrix has an inverse then it has a left inverse
and therefore the columns are linearly independent.

11) If the colums of a squre matrix A are linearly independent then A has a right inverse.

For a squre matrix A:

A has a left inverse $\Rightarrow$ the cols of A are linearly independent $\Rightarrow$ A has a right inverse.

$A = [a_1 \cdots a_n]$ each a_j is a vector in $\mathbb{R}^n$.

So $a_1, \ldots, a_n$ are n linearly independent vectors in $\mathbb{R}^n$,

They form a basis for $\mathbb{R}^n$.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Every vector in $\mathbb{R}^n$ can be written as a linear combination of $a_1, \dots, a_n$.

There exist numbers $\underbrace{b_{11}, \dots, b_{n1}}$ with $\underline{b_1} = \begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \end{bmatrix}$

$$b_{11} a_1 + b_{21} a_2 + \cdots + b_{n1} a_n = e_1$$

$$Ab_1 = e_1$$
$$Ab_2 = e_2$$
$$\vdots$$
$$Ab_n = e_n$$

$$B = [b_1 \ b_2 \cdots b_n]$$

Claim: B is a right inverse of A.

$$AB = A[b_1 \cdots b_n]$$
$$= [Ab_1 \ Ab_2 \cdots Ab_n]$$

$$= \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}$$

$$= I$$

12) For a square matrix A

A has a left inverse $\Rightarrow$ the cols of A are lin. ind. $\Rightarrow$ A has a right inverse

13) A has a right inverse $\Rightarrow$ the rows of A are lin. ind. $\Rightarrow$ A has a left inverse.

Suppose the rows of A are lin. ind.

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Then the columns of A^T are lin ind.

So A^T has a right inverse X.

$$A^T X = I$$

$$(A^T X)^T = I^T = I$$

$$\hookrightarrow = X^T (A^T)^T = X^T A$$

So X^T is a left inverse of A.

Big deal: The following are equivalent for a square matrix A.

invertible

- A has a left inverse
- A has a right inverse
- A has an inverse (A is invertible)
- the columns of A are linearly independent
- the rows of A are linearly independent.

Moreover, for a matrix satisfying one (and therefore all) of the above, $Ax = b$ always has a unique solution.

(In fact, if $Ax=b$ always has a unique solution then A is invertible)

14) If A and B are both $n \times n$ and invertible then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.

Why? $C = B^{-1}A^{-1}$ $\qquad ABB^{-1}A^{-1}$

$C \cdot (AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I.$

So C is a left inverse of AB.

15) If A is invertible so is A^T

and $(A^T)^{-1} = (A^{-1})^T$.

$$(A^{-1})^T A^T = (A A^{-1})^T = I^T = I$$

So $(A^{-1})^T$ is a left inverse of A^T and

hence is $(A^T)^{-1}$.

Examples: 1) I $\quad II = I$ ✓

2) $\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$ $d_i \neq 0$

is invertible

$$\begin{bmatrix} d_1^{-1} & & 0 \\ & d_2^{-1} & \\ & & \ddots & \\ 0 & & & d_n^{-1} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

3) Suppose A is an orthogonal matrix (square and has o.n. columns).

$A^T A = I \Rightarrow A^T$ is a left inverse of A

$\Rightarrow A^T = A^{-1}$