Solving Linear Systems

given $\qquad\qquad$ given

$$\underbrace{A}_{m \times n}\ \underbrace{x}_{\in \mathbb{R}^n} = \underbrace{b}_{\in \mathbb{R}^m}$$

$\boxed{m}$ equations for $\boxed{n}$ of unknowns

$m < n$, A wide, fewer equations than unknowns

underdetermined

$m > n$, A tall, more equations than unknowns

over-determined

$m = n$, A square, Goldilocks

same # of equations as unknowns

$m = n = 1$

$ax = b$

$a, b \in \mathbb{R}$

$\underbrace{\frac{1}{a} = a^{-1}}$

multiplicative inverse of a

$a^{-1}a = 1$

$\underbrace{a^{-1}a}x = a^{-1}b$

$1x = a^{-1}b$

$x = a^{-1}b$

$A^{-1}A = I$

A "A^{-1}"

Matrices have a notion of multiplicative inverse but we need to keep track of the difference between left and right.

Let A be a matrix.

We say X is a left inverse of A if

$$XA = I$$

We say X is a right inverse of A if

$$AX = I$$

A : $m \times n$ matrix.

$$XA = I$$

$$\boxed{n} \times m \qquad m \times n \qquad \boxed{n} \times \boxed{n}$$

$$\underset{X}{\overset{m}{n\begin{bmatrix}\quad & \quad & \quad\end{bmatrix}}}\;\underset{A}{\overset{n}{\begin{bmatrix}\quad\\ \quad\\ \quad\end{bmatrix}m}} = \overset{n}{\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}}n$$

Can X also be, for this full matrix, a right inverse?

$$m\overset{n}{\begin{bmatrix}\quad\\ \quad\\ \quad\end{bmatrix}}\;n\overset{m}{\begin{bmatrix}\quad & \quad & \quad\end{bmatrix}}$$

maybe??

but no guarantees!

Hope: If you can find a (left?, right?) inverse X

maybe the solution of $Ax = b$ is

simply Xb.

$$ax = b$$
$$x = a^{-1}b \quad ?$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

claim: X is a left inverse of A

$$X = \frac{1}{2}\begin{bmatrix} 0 & -6 & 4 \\ 0 & 5 & -3 \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 0 & -6 & 4 \\ 0 & 5 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$$

wanted a 1.

$$\frac{1}{2}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 0 & -6 & 4 \\ 0 & 5 & -3 \end{bmatrix} = \begin{bmatrix} 0 & & \\ & & \\ & & \end{bmatrix}$$

X is <u>not</u> a right inverse.

left inverses need not be right inverses, and vice versa.

$a = 5 \qquad a^{-1} = \frac{1}{5}$

$$\tilde{X} = \frac{1}{2}\begin{bmatrix} -4 & 2 & 0 \\ 3 & -1 & 0 \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} -4 & 2 & 0 \\ 3 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

A matrix can have more than one left inverse.

If a matrix has two different left inverses, then it has infinitely many:

$$\alpha + \beta = 1$$

$$\underbrace{\alpha X + \beta \tilde{X}}_{\to \text{ all left inverses}}$$

(HW)

We will say X is an inverse of A if

$$XA = I \quad \text{and} \quad AX = I$$

in which case we will write $X = A^{-1}$

isn't quite justified:
can there be
more than one inverse?

$$\begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix}$$

2×2 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

what could go wrong?

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$ad - bc = 0$

claim: this guy really is an inverse of A

$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \frac{1}{ad-bc}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \frac{1}{ad-bc} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$ad - bc$ is called the determinant of A $\underline{\underline{(2\times 2)}}$

$\underbrace{\text{the determinant of } A}_{\det(A)}$

If $\det(A) \neq 0$ then A has an inverse.

Converse is also true, but we haven't proven that yet.

Suppose A has an inverse $\underline{\underline{A^{-1}}}$.

We want to solve

$$Ax = b$$

<u>If a solution exists then</u>

$$\underbrace{A^{-1}A}_{I}x = A^{-1}b$$

A^{-1} is a left inverse

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

two possibilities: one solution

no solutions?

You give me b. I form $A^{-1}b$. Is this a solution?

$$A(A^{-1}b) = (AA^{-1})b$$

A^{-1} is a right inverse

$$= Ib = b \checkmark$$

yep!

Then $A^{-1}b$ is the one and only solution.

Upshot: If A has a two sided inverse then $Ax = b$ always has exactly one solution, and it is given by $x = A^{-1}b$.

From the above: if A has a left inverse X then the only possible solution of $Ax = b$ is $x = Xb$. However, it need not be true that Xb is actually a solution.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad X = \frac{1}{2}\begin{bmatrix} 0 & -6 & 4 \\ 0 & 5 & -3 \end{bmatrix}$$

$$b = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

$$Ax = b$$

$$Ax = b$$

The solution, if it exists, is

$$\frac{1}{2}\begin{bmatrix} 0 & -6 & 4 \\ 0 & 5 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$XAx = Xb$$

$$Ix = Xb$$

$$x = Xb$$

If there is an x with $Ax = b$ then $x = Xb$

Did it work?

b? No $b = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

X is a right inverse

want $Ax = b$

$$A(Xb) = (AX)b = Ib = b$$