Matrix Multiplication

Matrices with orthonormal columns

$A \quad m \times 3 \qquad A = \begin{bmatrix} \underset{\uparrow \, \mathbb{R}^m}{a_1} & a_2 & a_3 \end{bmatrix}$

$$\overbrace{A^T A}^{\text{Gram matrix}} = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & a_1^T a_3 \\ a_2^T a_1 & a_2^T a_2 & a_2^T a_3 \\ a_3^T a_1 & a_3^T a_2 & a_3^T a_3 \end{bmatrix}$$

The columns of A are orthonormal

$$\Longleftrightarrow \qquad a_i^T a_j = \begin{cases} 1 \; (\to \|a_i\|^2) & i = j \\ 0 & i \neq j \end{cases}$$

$$\iff A^T A = I$$

If A is square and has orthonormal columns we say A is an <u>orthogonal</u> matrix.

2x2 rotation matrices

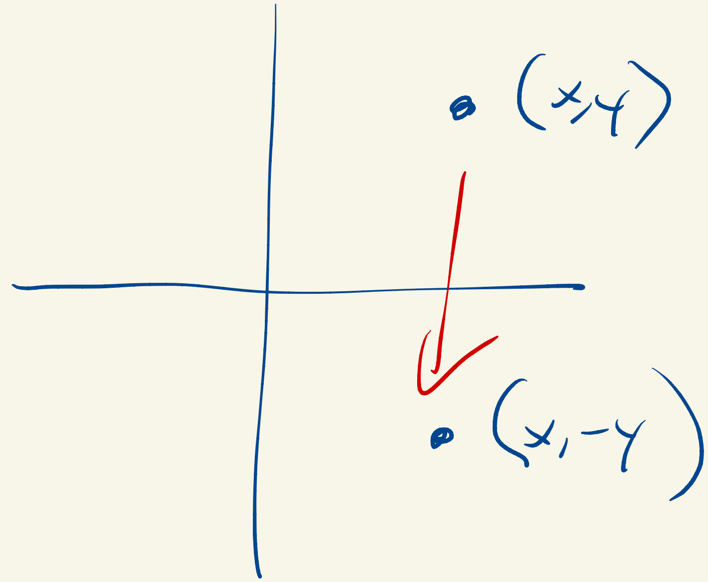
identity matrices

permutation matrices $\longrightarrow$ each row has one 1
each col has one 1
all others are 0's.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

reflection about x-axis



Geometric properties of matrices with orthonormal columns.

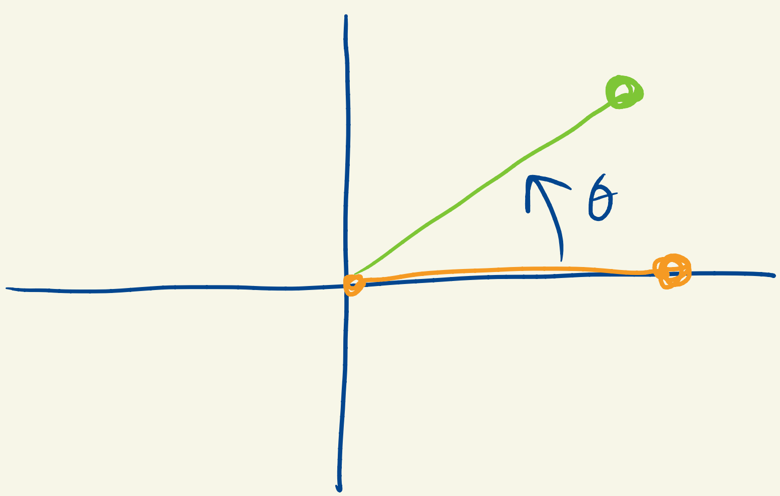
$$\underset{\substack{\uparrow \\ m \times n}}{A} \qquad f(x) = \overbrace{A \underset{\substack{\uparrow \\ \mathbb{R}^n}}{x}}^{\to\ \mathbb{R}^m}$$

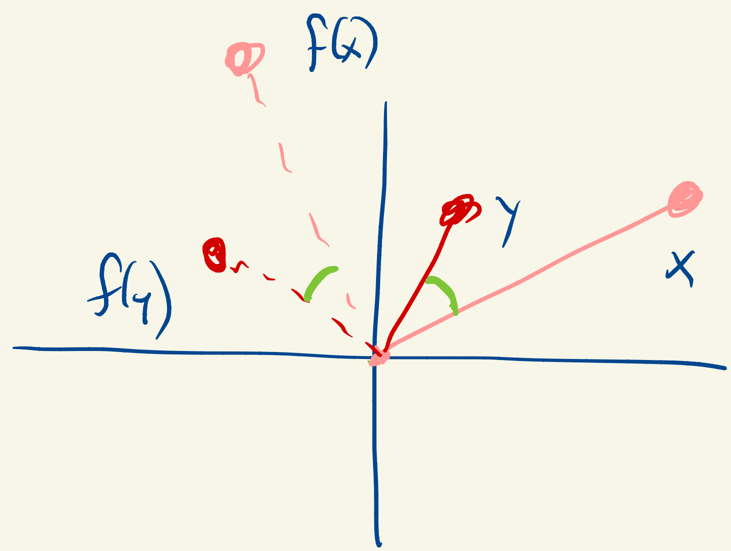
Claim (given A has o.n. columns)

a) for all $x \in \mathbb{R}^n$, $\|f(x)\| = \|x\|$

b) for all $x, y \in \mathbb{R}^n$ $f(x)^T f(y) = x^T y$

c) for all $x, y \in \mathbb{R}^n$ $\angle(f(x), f(y)) = \angle(x, y)$





$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{f(x)}$$

$\|x\| = 1$ $\qquad \|f(x)\| = \sqrt{5}$

$(AB)^T = B^T A^T$

b) $x, y \in \mathbb{R}^n$

$$f(x)^T f(y) = \overbrace{(Ax)^T}(Ay)$$

$$= (x^T A^T)(Ay)$$

$$= x^T (A^T A) y$$

$$= x^T I y$$

$$= x^T y$$

a) $\|f(x)\|^2 = f(x)^T f(x) = x^T x = \|x\|^2$

c) $\measuredangle(x,y) = \arccos\left(\frac{x^T y}{\|x\|\,\|y\|}\right)$

$$= \arccos\left(\frac{f(x)^T f(y)}{\|f(x)\|\,\|f(y)\|}\right)$$

$$= \measuredangle(f(x), f(y))$$

$$A = Q\,R$$

Q: orthonormal cols

R: upper triangular

"QR factorization"

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scales rows $\Rightarrow$

$$\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} = \begin{bmatrix} 3c_1 & 3c_2 \\ 7d_1 & 7d_2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 3a_1 & 7b_1 \\ 3a_2 & 7b_2 \end{bmatrix}$$

scales columns

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 3a_1 & 2a_1 + 7b_1 \\ 3a_2 & 2a_2 + 7b_2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \end{bmatrix} \qquad = \begin{bmatrix} 3a & 2a + 7b \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} = \begin{bmatrix} 3c_1 + 2d_1 & 3c_2 + 2d_2 \\ 7d_1 & 7d_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} c^T \\ d^T \end{bmatrix} = \begin{bmatrix} 3c^T + 2d^T \\ 7d^T \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \underbrace{b_{11} a_1} & \underbrace{b_{12} a_1 + b_{22} a_2} & \underbrace{b_{13} a_1 + b_{23} a_2 + b_{33} a_3} \end{bmatrix}$$

Gram Schmidt

$\tilde{q}_i \qquad \tilde{q}_i$

$a_1 \qquad a_2 \qquad a_3$

$\tilde{q}_1 = a_1 \qquad\qquad q_1 = \tilde{q}_1 / \|\tilde{q}_1\|$

$\tilde{q}_2 = a_2 - (q_1^T a_2) q_1 \longrightarrow q_1^T \tilde{q}_2 = (q_1^T a_2) - (q_1^T a_2)\underbrace{(q_1^T q_1)}_{1}$

$q_2 = \tilde{q}_2 / \|\tilde{q}_2\|$

$= 0$

$$\tilde{q}_3 = a_3 - (q_1^T a_3)\, q_1 - (q_2^T a_3)\, q_2$$

$$q_3 = \tilde{q}_3 / \|q_3\|$$

$$q_1 = \tilde{q}_1 / \|\tilde{q}_1\|$$

$$\tilde{q}_1 = \|\tilde{q}_1\|\, q_1$$

$$a_1 = \tilde{q}_1$$

$$a_2 = \tilde{q}_2 + (q_1^T a_2)\, q_1$$

$$a_3 = \tilde{q}_3 + (q_1^T a_3)\, q_1 + (q_2^T a_3)\, q_2$$

$$a_1 = \|\tilde{q}_1\| \, q_1$$

$$a_2 = (q_1^T a_2) \, q_1 + \|\tilde{q}_2\| \, q_2$$

$$a_3 = (q_1^T a_3) \, q_1 + (a_2^T a_3) \, q_2 + \|\tilde{q}_3\| \, q_3$$

$$\underset{Q}{\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}} \underbrace{\begin{bmatrix} \|\tilde{q}_1\| & q_1^T a_2 & q_1^T a_3 \\ 0 & \|\tilde{q}_2\| & q_2^T a_3 \\ 0 & 0 & \|\tilde{q}_3\| \end{bmatrix}}_{R}$$

$$= \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

Given a matrix A with linearly independent columns we can factor

$$A = QR$$

where Q has orthonormal cols

R is upper tri and has nonzero diagonal entries

Why care?

want to solve for x

$$Ax = b$$

$$\downarrow$$

$$QRx = b$$

$$Q^TQRx = Q^Tb$$

$$Rx = Q^Tb$$

R is upper triangular

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 19 \end{bmatrix}$$

$$x_3 = 19/6$$

It's easy to solve $Rx = Q^T b$ for x.

$$Ax = b$$