$x \longrightarrow Ax$

representing linear maps

$x \longmapsto c^T x$

AB

$$\overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}} = \begin{bmatrix} 14 & 0 \\ 32 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\parallel$$

$$1 \cdot \begin{bmatrix} 1 & -2 \end{bmatrix} + 2 \begin{bmatrix} 2 & 1 \end{bmatrix} + 3 \begin{bmatrix} 3 & 0 \end{bmatrix}$$

$$3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 32 & -3 \end{bmatrix}$$

$$r_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad r_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} r_1^T \\ r_2^T \end{bmatrix}$$

$$AB = \begin{bmatrix} r_1^T B \\ r_2^T B \end{bmatrix} \begin{matrix} \leftarrow \begin{bmatrix} 14 & 0 \end{bmatrix} \\ \leftarrow \begin{bmatrix} 32 & -3 \end{bmatrix} \end{matrix}$$

In general

$$A = \begin{bmatrix} r_1^T \\ \vdots \\ r_k^T \end{bmatrix} B = \begin{bmatrix} r_1^T B \\ \vdots \\ r_k^T B \end{bmatrix}$$

$c^T x$

$$\begin{bmatrix} 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = -3 + 2 - 4 = -5$$

inner product

outer product $\rightarrow$

$$\underset{3 \times 1}{\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}} \underset{1 \times 3}{\begin{bmatrix} 3 & 1 & 2 \end{bmatrix}} = \begin{bmatrix} -3 & -1 & -2 \\ 6 & 2 & 4 \\ -6 & -2 & -4 \end{bmatrix}$$

A $\quad m \times n$ $\qquad A = [a_1\ a_2 \cdots a_n]$

$$\underset{m \times m}{I}\ \underset{m \times n}{A} = I\,[a_1 \cdots a_n] = [Ia_1\ Ia_2 \cdots Ia_n]$$

$$= [a_1\ a_2 \cdots a_n]$$

$$= A$$

$$\underset{m \times n}{A}\ \underset{n \times n}{I} = \begin{bmatrix} r_1^T \\ \vdots \\ r_m^T \end{bmatrix} I$$

$$= \begin{bmatrix} r_1^T I \\ \vdots \\ r_m^T I \end{bmatrix}$$

$$= \begin{bmatrix} r_1^T \\ \vdots \\ r_n^T \end{bmatrix}$$

$$= A$$

$$A = \begin{bmatrix} r_1^T \\ \vdots \\ r_m^T \end{bmatrix}$$

$$\begin{bmatrix} 3 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 & 7 \end{bmatrix}$$

$$\overset{A}{\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}} \overset{B}{\begin{bmatrix} 4 & 0 \\ 5 & 6 \end{bmatrix}} = \overset{AB}{\begin{bmatrix} 4 & 0 \\ 23 & 18 \end{bmatrix}}$$

$$\overset{B^T}{\begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}} \overset{A^T}{\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}} = \overset{(AB)^T}{\begin{bmatrix} 4 & 23 \\ 0 & 18 \end{bmatrix}}$$

$n \times m \quad k \times n$

$$(AB)^T = B^T A^T \qquad \cancel{(AB)^T = A^T B^T}$$

$m \times n \quad n \times k$

$A \quad m \times n \qquad B \quad n \times k$

$$(AB)_{ij} = \sum_{l=1}^{n} A_{il} B_{lj}$$

$$(AB)^T_{ij} = (AB)_{ji} = \sum_{l=1}^{n} A_{jl} B_{li}$$

$$= \sum_{l=1}^{n} (A^T)_{lj} (B^T)_{il}$$

$$= \sum_{l=1}^{n} (B^T)_{il} (A^T)_{lj}$$

$$= (B^T A^T)_{ij}$$

$$(AB)^T = B^T A^T$$

$$\begin{array}{c|cc} & n_1 & n_2 \\ \hline m_1 & A_1 & B_1 \\ m_2 & C_1 & D_1 \end{array} \quad \begin{array}{c|cc} & k_1 & k_2 \\ \hline n_1 & A_2 & B_2 \\ n_2 & C_2 & D_2 \end{array}$$

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} \overset{m_1 \times k_1}{(A_1 A_2 + B_1 C_2)} & \overset{m_1 \times k_2}{(A_1 B_2 + B_1 D_2)} \\ \underset{m_2 \times k_1}{(C_1 A_2 + D_1 C_2)} & \underset{m_2 \times k_2}{(C_1 B_2 + D_1 D_2)} \end{bmatrix}$$

$A_1 : m_1 \times n_1$

Block multiplication works

$A \quad m \times n$

$$\underset{n \times m}{A^T} \; \underset{m \times n}{A}$$

$$\underset{m \times n}{A} \; \underset{n \times m}{A^T}$$

$$\overset{m \times n}{A} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}$$

$$A^T A = \begin{bmatrix} a_1^T \\ \vdots \\ \vdots \\ a_n^T \end{bmatrix} \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$$

Gram matrix of A

$$= \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \cdots & a_1^T a_n \\ a_2^T a_1 & a_2^T a_2 & \cdots & a_2^T a_n \\ \vdots & & & \\ a_n^T a_1 & \cdots & \cdots & a_n^T a_n \end{bmatrix}$$

$n \times n$

$$(A^T A)_{ij} = (A^T A)_{ji}$$

$$C_{ij} = C_{ji}$$

"symmetric matrix"

$$C^T = C$$

square

$$(A^T A)_{ij} = a_i^T a_j$$

dot product of column i with column j of A

$(AA^T)_{ij}$ is row$_i$ of A dotted with row$_j$ of A.

$m \times m$

$$Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \qquad O$$

q_j

↳ orthonormal

$$Q^TQ = \begin{bmatrix} q_1^Tq_1 & q_1^Tq_2 & \cdots & q_1^Tq_n \\ q_2^Tq_1 & q_2^Tq_2 & \cdots & q_2^Tq_n \\ \vdots & & & \\ q_n^Tq_1 & - & - & q_n^Tq_n \end{bmatrix}$$

$$q_j^Tq_i = \begin{cases} 0 & i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$= I$$

$$Qx = b$$

$$Q^TQx = Q^Tb$$

$$Ix = Q^Tb$$

$$x = Q^Tb$$