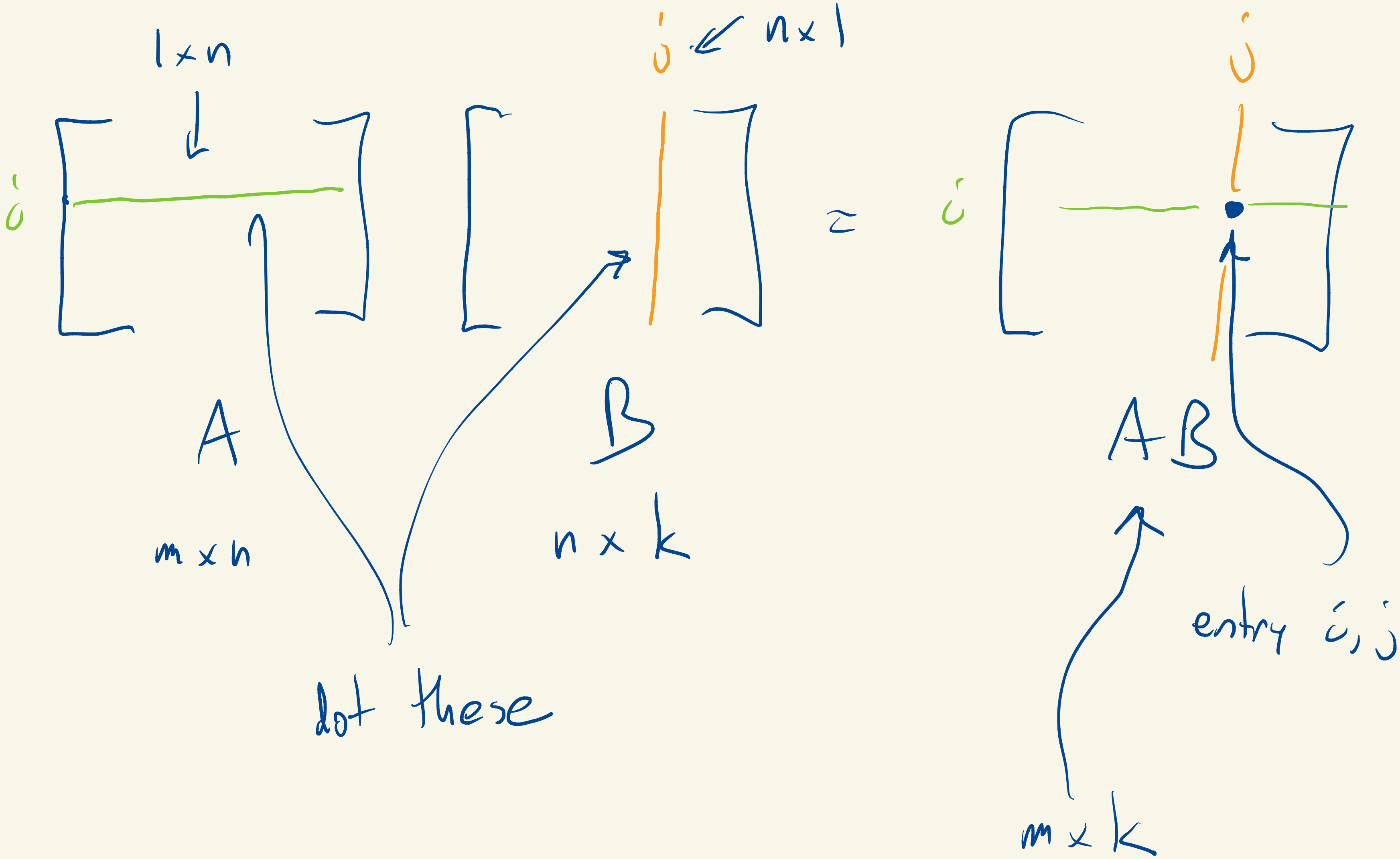
Matrix Matrix multiplication

first row, second col, first row second col

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}} = \begin{bmatrix} 14 & 0 \\ 4\cdot 1+5\cdot 2+6\cdot 3 & -3 \end{bmatrix}$$

$$\boxed{2} \times 3 \qquad 3 \times \boxed{2} \rightarrow \boxed{2} \times \boxed{2}$$



$1 \times n$
i
j
$n \times 1$
$=$
i
j
A
$m \times n$
B
$n \times k$
AB
entry i, j
dot these
$m \times k$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{array}{c} \begin{array}{cc} b_1 & b_2 \\ \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 32 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 14 & 0 \\ 32 & -3 \end{bmatrix}$$

$$AB = A \begin{bmatrix} b_1 & b_2 \end{bmatrix}$$
$$= \begin{bmatrix} Ab_1 & Ab_2 \end{bmatrix}$$

"Column Perspective" of matrix-matrix multiplication.

In general $\underset{m \times n}{A}$ $\underset{n \times k}{B}$ $B = [b_1, b_2, \dots, b_k]$

$$AB = [Ab_1 \ \ Ab_2 \ \ \dots\dots \ \ Ab_k]$$

$$m \times k$$

The ℓ^{th} column of AB is A times the ℓ^{th} column of B

Why? What's the point?

1) We represent linear maps $f : \mathbb{R}^n \to \mathbb{R}^m$ in terms of matrix-vector multiplication.

There is a $m \times n$ Matrix A such that $f(x) = Ax$ for all x in $\mathbb{R}^n$.

$$A = \left[f(e_1) \ f(e_2) \cdots f(e_n) \right]$$

2) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^k$ are linear

There are matrices A $m \times n$

B $k \times m$

$f(x) = Ax$

$g(y) = By$

Consider $g(f(x))$ $(g \circ f)(x)$

$A = [a_1, \ldots, a_n]$ $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$B = [b_1 \ldots b_m]$

$$f(x) = Ax = [a_1 \cdots a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

$$g(f(x)) = g(x_1 a_1 + \cdots + x_n a_n)$$

$$= x_1 g(a_1) + x_2 g(a_2) + \cdots + x_n g(a_n)$$

$$= x_1 \underbrace{Ba_1}_{\text{k vector}} + x_2 \underbrace{Ba_2}_{\text{k veck}} + \cdots + x_n \underbrace{Ba_n}_{\text{k vector}}$$

$$= \begin{bmatrix} Ba_1 & Ba_2 & \cdots & Ba_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= BAx$$

Upshot: If $f(x) = Ax$

$g(y) = By$

$g \circ f$ makes sense

Then $(g \circ f)(x) = BAx$

Matrix multiplication corresponds with composition of functions

$$f, g : \mathbb{R}^4 \to \mathbb{R}^4 \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$f(x) = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \\ x_4 \end{bmatrix} \qquad g(y) = \begin{bmatrix} y_1 \\ y_4 \\ y_3 \\ y_2 \end{bmatrix}$$

$$\begin{tikzcd}
(x_1, x_2, x_3, x_4) \arrow[r, "f"] \arrow[rr, bend right, "(g \circ f)(x)"'] & (x_2, x_1, x_3, x_4) \arrow[r, "g"] & (x_2, x_4, x_3, x_1)
\end{tikzcd}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$f(x) = Ax$$

$$A = \begin{array}{cccc} \left[\begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right] \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ f(e_1)\ f(e_2)\ f(e_3)\ f(e_4) \end{array}$$

$$g(y) = By$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$g(f(x)) = g(Ax) = BAx$$

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \\ x_4 \end{bmatrix} = f(x)$$

$$\overset{A}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underset{B}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} = \underset{BA}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}$$

$$BA \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \\ x_3 \\ x_1 \end{bmatrix}$$

$$Bx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} x_2 \\ x_4 \\ x_3 \\ x_1 \end{bmatrix}$$

$$(x_2, x_4, x_3, x_1)$$

$$
\begin{tikzcd}
(x_1, x_2, x_3, x_4) \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & (x_2, x_1, x_3, x_4) \arrow[r, "g"] & (x_2, x_4, x_3, x_1) \arrow[d, leftrightarrow, "\text{not equal}"] \\
(x_1, x_2, x_3, x_4) \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & (x_1, x_4, x_3, x_2) \arrow[r, "f"] & (x_4, x_1, x_3, x_2)
\end{tikzcd}
$$

$$g \circ f \neq f \circ g$$

$$BA \neq AB$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 0 \\ 6 & 4 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 0 \\ 8 & 4 \end{bmatrix}$$

"matrix multiplication is not commutative"

Good news

$$A(B+C) = AB + AC$$

$$A(BC) = (AB)C \qquad ABC$$

$$(\gamma A)C = A(\gamma C) = \gamma AC$$