Gram Schmidt

Idea: start with a collection of vectors

a_1, a_2, a_3

We're going to build some orthonormal vectors

q_1, q_2, q_3

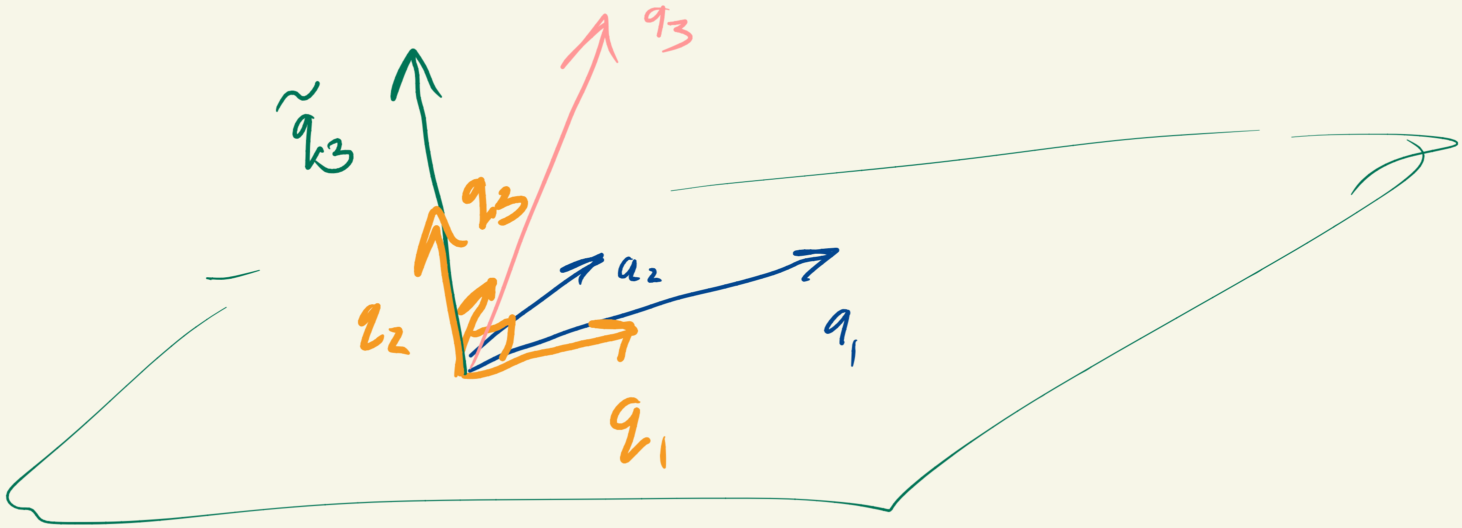
With the following properties:

q_1 is just a multiple of a_1

q_2 is a linear combination of a_1 and a_2

q_3 is a linear combination of a_1, a_2 and a_3

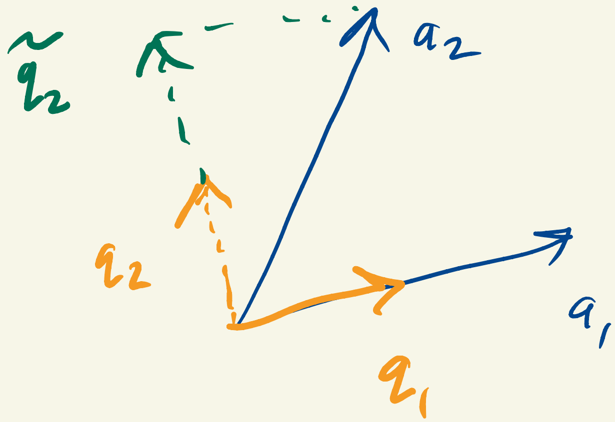
(And vice-versa! a_3 is a linear combo of q_1, q_2, q_3)



a_1 , a_2 , q_3

$\tilde{q}_1 = a_1$

$q_1 = \tilde{q}_1 / \|\tilde{q}_1\|$ ⟵ unit vector.



$$a_2 = \tilde{q}_2 + c\,q_1$$

$$\tilde{q}_2 = a_2 - \underset{\substack{\uparrow \\ \text{unknown.}}}{c}\,q_1$$

goal $\tilde{q}_2 \perp q_1$

Want $q_1^T \tilde{q}_2 = 0$

$$q_1^T (a_2 - c\,q_1) = q_1^T a_2 - c\,q_1^T q_1$$

$$= q_1^T a_2 - c$$

$$c = \underbrace{q_1^T a_2}$$

"How much of a_2 is pointing along q_1?"

$$q_2 = \tilde{q}_2 / \|\tilde{q}_2\|$$

goal: $\tilde{q}_3 \perp q_2$

$\tilde{q}_3 \perp q_1$

$\tilde{q}_3^T q_2 = 0$

$q_1^T \tilde{q}_3 = 0$

$$\tilde{q}_3 = a_3 - \underbrace{c_1 q_1 - c_2}_{\text{unknown}} q_2$$

$$\underbrace{q_1^T \tilde{q}_3}_{\text{want} = 0} = q_1^T (a_3 - c_1 q_1 - c_2 q_2)$$

$$= q_1^T a_3 - c_1 q_1^T q_1 - c_2 \underbrace{q_1^T q_2}_{0}$$

$$= q_1^T a_3 - c_1$$

$$c_1 = q_1^T a_3$$

$$c_2 = q_2^T a_3$$

$$q_3 = \tilde{q}_3 / \|\tilde{q}_3\|$$

$$\tilde{q}_3 = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$$

a_4

$$\tilde{q}_4 = a_4 - (q_1^T a_4) q_1 - (q_2^T a_4) q_2 - (q_3^T a_4) q_3$$

$$q_4 = \tilde{q}_4 / \|\tilde{q}_4\|$$

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

$\|a_1\| = \sqrt{6}$

$$q_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\tilde{q}_2 = a_2 - \underbrace{(q_1^T a_2)}\, q_1 \qquad q_1^T a_2 = \frac{1}{\sqrt{6}}\left(4 - 2 + 3\right) = \frac{5}{\sqrt{6}}$$

$$\tilde{q}_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 19/6 \\ -16/6 \\ 13/6 \end{bmatrix}$$

$$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{1}{\sqrt{19^2 + 16^2 + 13^2}}\begin{bmatrix} 19 \\ -16 \\ 13 \end{bmatrix}$$

Is $q_2 \perp q_1$?

$$\begin{bmatrix} 19 \\ -16 \\ 13 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 19 - 2\cdot 16 + 13$$

$$= 32 - 32 = 0$$

$$\tilde{q}_k = a_k - (q_1^T a_k) q_1 - \cdots - (q_{k-1}^T a_k) q_{k-1}$$

$$q_k = \tilde{q}_k / \|\tilde{q}_k\|$$

Rinse + repeat.

$$a_k = \beta_1 q_1 + \cdots + \beta q_{k-1}$$

$q_1, \ldots, q_{k-1}$ are linear combos of $a_1, \ldots, a_{k-1}$

$a_1, a_2, \ldots, a_{k-1}, a_k$ is not linearly independent.

Upshot: If all the a's are linearly independent we never have a $\tilde{q}_k = 0$ so we never risk dividing by zero.

The algorithm generates a full set of k's.

Converse: If one a_k is a linear combo of $a_1, \ldots, a_{k-1}$ one can show that $\tilde{q}_k = 0$.

If the a's are linearly dependent, we can test this with Gram Schmidt:

one $\tilde{q}_k$ will be zero ~~iff~~ the a's are linearly dependent.

$5/7 - \frac{1}{7} \cdot 5$ 10^{-16}