$$= \left[\frac{n}{n}\right]^{1/2} = \sqrt{1} = 1$$

$$X = (x_1, x_2, x_3, \ldots, x_n)$$

$\longrightarrow$ returns on an investment for each time period.

avg(x) "expected return on investment in a single period"

std(x) "expected deviation from the average"

$\longrightarrow$ risk of investment

$$\mathrm{std}(x) = \mathrm{rms}\left(x - \mathrm{avg}(x)\mathbf{1}\right) \qquad \mathrm{rms}(y) = \frac{\|y\|}{\sqrt{n}}$$

$$\mathrm{std}(x)^2 = \mathrm{rms}\left(x - \mathrm{avg}(x)\mathbf{1}\right)^2 \qquad y^T y = \|y\|^2$$

$$= \frac{\|x - \mathrm{avg}(x)\mathbf{1}\|^2}{n}$$

$$= \frac{1}{n}\left[(x - \mathrm{avg}\, x\mathbf{1})^T (x - \mathrm{avg}(x)\mathbf{1}) \right]$$

$$= \frac{1}{n}\left[x^T x - \mathrm{avg}\, x\, x^T\mathbf{1} - \mathrm{avg}\, x\, \mathbf{1}^T x + \mathrm{avg}(x)^2 \mathbf{1}^T\mathbf{1} \right]$$

$$= \frac{1}{n}\left[\|x\|^2 - 2\,\mathrm{avg}(x)\,\underbrace{x^T\mathbf{1}} + \mathrm{avg}(x)^2 \cdot n \right]$$

$$= \frac{1}{n}\left[\|x\|^2 - 2\,\mathrm{avg}(x)\, n\,\mathrm{avg}(x) + n\,\mathrm{avg}(x)^2 \right]$$

$$= \frac{1}{n}\left[\|x\|^2 - n\,\mathrm{avg}(x)^2 \right]$$

$$x^T\mathbf{1} = x_1 \cdot 1 + x_2 \cdot 1 + \dots + x_n \cdot 1$$

$$= x_1 + x_2 + \dots + x_n$$

$$= n\,\mathrm{avg}(x)$$

$$= \frac{\|x\|^2}{n} - \mathrm{avg}(x)^2$$

$$= \mathrm{rms}(x)^2 - \mathrm{avg}(x)^2$$

$$\mathrm{std}(x)^2 = \mathrm{rms}(x)^2 - \mathrm{avg}(x)^2$$

$$\boxed{\mathrm{rms}(x)^2 = \mathrm{avg}(x)^2 + \mathrm{std}(x)^2}$$

Angles between vectors.

Cauchy-Schwartz Inequality

$x, y \in \mathbb{R}^n$

$$|x^T y| \leq \|x\| \, \|y\|$$ $<$ (strict if x and y are not parallel)

We can prove the $\underbrace{\text{triangle inequality}}$; $\|x+y\| \leq \|x\| + \|y\|$

$$\|x+y\|^2 = (x+y)^T(x+y)$$
$$= \|x\|^2 + 2x^Ty + \|y\|^2$$

C-S inequality

$$\leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$

$$= \left(\|x\| + \|y\|\right)^2$$

$$\|x+y\| \leq \|x\| + \|y\|$$

Suppose u, v are vectors and $\|u\| = \|v\| = 1$

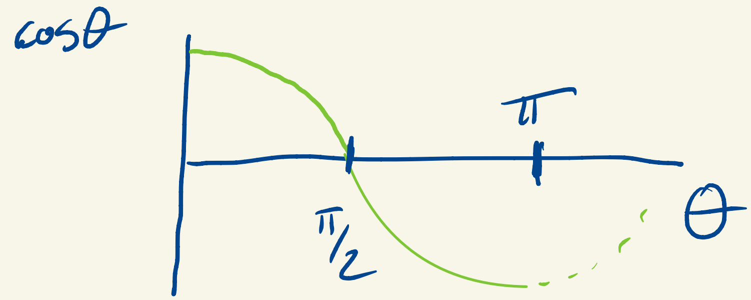
"unit vectors"

$$-1 \leq u^T v \leq 1$$

$$|u^T v| \leq \|u\| \cdot \|v\| = 1 \cdot 1 = 1$$

$$\angle(u,v) = \arccos(u^T v)$$

"angle between u and v"

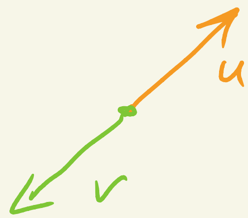


$u = v$ $\qquad$ $u^T \cdot v = v^T v = \|v\|^2 = 1$

$$\arccos(u^T v) = \arccos(1) = 0$$

$$\angle(u,v) = 0$$

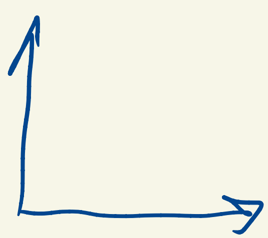
$u = -v$



$$u^T v = (-v)^T v$$
$$= -1\, v^T v$$
$$= -1 \cdot \|v\|^2$$
$$= -1$$

$\arccos(-1) = \pi$ $\qquad$ $\angle(u,v) = \pi \quad (180^\circ)$

$v = (0,1)$



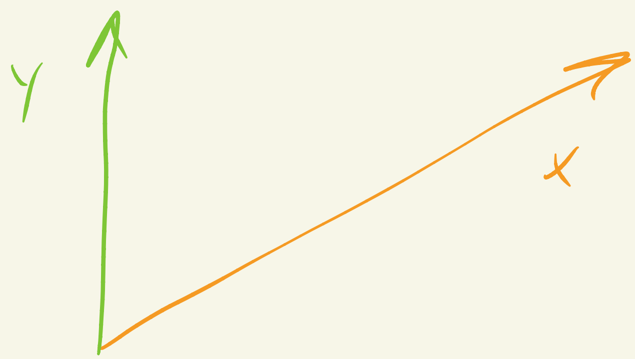
$u = (1,0)$

$$\angle(u,v) = \pi/2$$

$$u^T v = 1 \cdot 0 + 0 \cdot 1 = 0$$

$$\arccos(0) = \pi/2$$

What about arbitrary vectors x, y $\quad x \neq 0 \quad y \neq 0$



$$u = \frac{x}{\|x\|} \qquad v = \frac{y}{\|y\|}$$

$$\|u\| = \left\| \frac{x}{\|x\|} \right\|$$

$$= \left| \frac{1}{\|x\|} \right| \|x\|$$

$$= \frac{\|x\|}{\|x\|} = 1$$

$$\angle(x, y) = \angle(u, v)$$

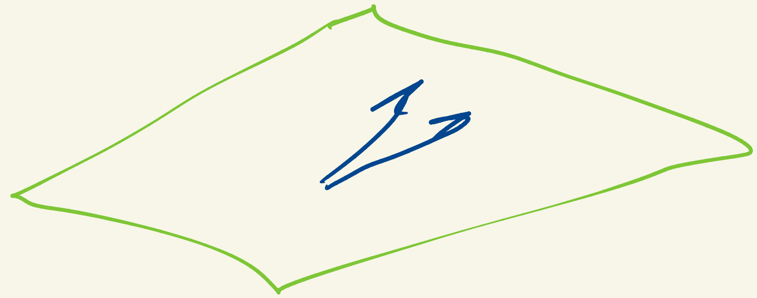
$$\cos(\theta) = \left(\frac{x}{\|x\|}\right)^T \frac{y}{\|y\|}$$

$$= \frac{x^T y}{\|x\| \, \|y\|}$$

$$x^T y = \|x\| \|y\| \cos(\theta)$$

$x = (1, 2, 1, -2)$

$y = (4, 1, 3, 2)$



What is the angle between x and y?

$$\theta = \arccos\left(\frac{1}{2\sqrt{3}} \right)$$