MATH 253
Quiz 11

Name: SOLUTIONS

April 8, 2022

Instructions: (10 points total – 5 pts each) Show all work for credit. You may use your book, but no other resource.

1. In this problem you will show that the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$$

for the vector field $\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ and C any positively-oriented, simple, closed circle enclosing the origin. Note that the vector field $\mathbf{F}$ is not defined at the origin, so the domain is the punctured plane.

(1pt) (a) Let $C = C_R$ denote the circle of radius R where $R > 0$. Give a parameterization $\mathbf{r}(t)$ for this circle of radius R, where the circle is traversed in the counter-clockwise direction starting and ending at $(R, 0)$.

Answer: $\mathbf{r}(t) = \langle R\cos t, R\sin t \rangle \qquad 0 \le t \le 2\pi$

(4pts) (b) Using your parameterization, compute the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

$$\vec{F}(\vec{r}(t)) = \left\langle \frac{-R\sin t}{R^2}, \frac{R\cos t}{R^2} \right\rangle = \left\langle -\frac{\sin t}{R}, \frac{\cos t}{R} \right\rangle$$

$$d\vec{r} = \vec{r}\,'(t)\,dt = \langle -R\sin t, R\cos t \rangle\,dt$$

Thus,

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \left\langle -\frac{\sin t}{R}, \frac{\cos t}{R} \right\rangle \cdot \langle -R\sin t, R\cos t \rangle\,dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t \;dt = \boxed{2\pi}$$

2. Consider the two dimensional vector field

$$\mathbf{F}(x,y) = \left\langle e^{xy}(y\sin(x)+\cos(x)),\ xe^{xy}\sin(x)+\frac{1}{y}\right\rangle$$

defined on the upper half plane in $\mathbb{R}^2$ (i.e. $y>0$)

(4pts)

(a) Prove that $\mathbf{F}$ is conservative on this open, simply-connected domain, then find its potential function $f(x,y)$.

Since the domain $y>0$ is open, simply-connected it suffices to check

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$: $P = e^{xy}(y\sin x + \cos x) \Rightarrow \frac{\partial P}{\partial y} = e^{xy}(\sin x) + xe^{xy}(y\sin x+\cos x)$

$= \underline{e^{xy}(\sin x + xy\sin x + x\cos x)}$

$Q = xe^{xy}\sin(x) + \frac{1}{y} \Rightarrow \frac{\partial Q}{\partial x} = xe^{xy}(\cos(x)) + [xye^{xy} + (1)e^{xy}]\sin(x)$

$= \underline{e^{xy}[x\cos x + xy\sin x + \sin x]}$ ✓ $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

For the potential function, start with $\int \frac{\partial f}{\partial y}\,dy = \int xe^{xy}\sin(x) + \frac{1}{y}\,dy$

$\Rightarrow f(x,y) = e^{xy}\sin(x) + \ln|y| + c(x)$ ← function of x alone.

Thus, $P = \frac{\partial f}{\partial x} = e^{xy}\cos(x) + ye^{xy}\sin(x) + c'(x) = e^{xy}(y\sin x + \cos(x))$

↑ by differentiating

↑ by $P = \frac{\partial f}{\partial x}$

Thus, $c'(x) = 0$ and c is a constant!

Thus, the potential function is

$\boxed{f(x,y) = e^{xy}\sin x + \ln|y| + C}$

(b) Letting C be the line segment joining $(0,1)$ to the point $(0,\frac{\pi}{2})$, compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$.

end pt. beginning pt.

(b) Easiest solution $\int_C \vec{F}\cdot d\vec{r} = f(0,\frac{\pi}{2}) - f(0,1)$

$= [e^0\sin 0 + \ln|\frac{\pi}{2}|] - [e^0\sin(0) + \ln|1|]$

$= \boxed{\ln \pi/2}$

1pt → No partial credit.