Name: Solutions

Stokes's Theorem: If C is the boundary of a 'nice' region $\mathcal{S}$,

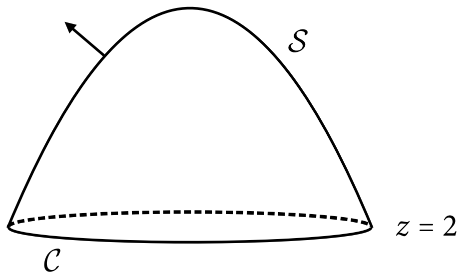
$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot d\mathbf{r}$$

so long as the normal $\mathbf{n}$ and the orientation of C are compatible.

In the two problems below you will set up, but **not evaluate** the integrals on both side of this equation where $\mathcal{S}$ is the **paraboloid** $z = 4 - x^2 - y^2$ with $z \geq 2$ and where

$$\mathbf{F} = \langle z, xy, y \rangle .$$

The surface is given the orientation with unit normal pointing n the direction given in the figure (generally in the positive z direction.)



1. Write down an integral expressing $\int_C \mathbf{F} \cdot d\mathbf{r}$. Your answer should be of the form $\int_a^b g(t)\, dt$ where a and b are numbers and where $g(t)$ is an explicit function. Please do not compute the integral. Please be careful about orientation/sign.

On boarday, $x^2+y^2 = 4-2 = 2$.

$\vec{\sigma}(t) = \langle \sqrt{2}\cos t, \sqrt{2}\sin t, 2 \rangle$

$\vec{\sigma}'(t) = \langle -\sqrt{2}\sin t, \sqrt{2}\cos t, 0 \rangle$

$\vec{F}(\vec{\sigma}(t)) = \langle 2, 2\cos t \sin t, \sqrt{2}\sin t \rangle$

$\vec{F} \cdot \vec{\sigma}' = -2\sqrt{2}\sin t + 2\sqrt{2}\cos(t)\sin^2(t)$

$$\boxed{\int_0^{2\pi} \left(-2\sqrt{2}\sin t + 2\sqrt{2}\cos(t)\sin^2(t)\right) dt}$$

Recall: $\mathcal{S}$ is the paraboloid $z = 4 - x^2 - y^2$ with $z \geq 2$ and unit normal pointing generally in the z direction and that

$$\mathbf{F} = \langle z, xy, y \rangle .$$

2. Write down an integral expressing $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$. Your answer should be in the form of an iterated integral of an explicit integrand that is a function of two parameter variables. The endpoints of each integral in the iterated integral must be explicit numbers or functions of parameter variables. Please do **not** compute the integral.

$$\vec{\nabla} \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ z & xy & y \end{vmatrix} = \langle 1, 1, y \rangle$$

$$\vec{r}(u,v) = \langle u, v, 4 - u^2 - v^2 \rangle$$

$$\vec{r}_u = \langle 1, 0, -2u \rangle$$

$$\vec{r}_v = \langle 0, 1, -2v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 2u, 2v, 1 \rangle$$

$$(\vec{\nabla} \times \vec{F}) \cdot \vec{r}_u \times \vec{r}_v = 2u + 2v + v$$

$$\boxed{\int_{-2}^{2} \int_{-\sqrt{4-u^2}}^{\sqrt{4-u^2}} (2u + 3v)\, dv\, du}$$

Recall: $\mathcal{S}$ is the paraboloid $z = 4 - x^2 - y^2$ with $z \geq 2$ and unit normal pointing generally in the z direction and that

$$\mathbf{F} = \langle z, xy, y \rangle .$$

2. Write down an integral expressing $\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$. Your answer should be in the form of an iterated integral of an explicit integrand that is a function of two parameter variables. The endpoints of each integral in the iterated integral must be explicit numbers or functions of parameter variables. Please do **not** compute the integral.

Att: $\vec{r}(\rho,\theta) = \langle \rho\cos\theta, \rho\sin\theta, 4-\rho^2 \rangle$

$\vec{r}_\rho = \langle \cos\theta, \sin\theta, -2\rho \rangle$

$\vec{r}_\theta = \langle -\rho\sin\theta, \rho\cos\theta, 0 \rangle$

$\vec{r}_\rho \times \vec{r}_\theta = \langle 2\rho^2\cos\theta, 2\rho^2\sin\theta, \rho \rangle$

$\vec{\nabla}\times\vec{F} = \langle 1, 1, y \rangle$, $(\vec{\nabla}\times F)(\vec{r}(\rho,\theta)) = \langle 1, 1, \rho\sin\theta \rangle$

$\vec{\nabla}\times\vec{F}\cdot(\vec{r}_u\times\vec{r}_v) = 2\rho^2\cos\theta + 2\rho^2\sin\theta + \rho^2\sin\theta$

$$\boxed{\int_0^{2\pi}\int_0^{\sqrt{2}} 2\rho^2\cos\theta + 3\rho^2\sin\theta \quad d\rho\, d\theta}$$