MATH 253 Name: ______________________________

Midterm 2 March 23, 2021

Instructions: 100 points total. Use only your brain and writing implement. You have 90 minutes to complete this exam. Good luck.

1. (8 pts.) Prove that the following limit does **NOT** exist.

$$\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^4+y^2}$$

Sol. Let $f(x,y) = \frac{4x^2y}{x^4+y^2}$.

If $y = 0$, then $f(x,0) = \dfrac{0}{x^4} = 0$. Therefore $f(x,y) \to 0$ as $(x,y) \to 0$ along the $x-$ axis.

If $y = x^2$, then $f(x,x^2) = \dfrac{4x^4}{x^4+x^4} = \dfrac{4x^4}{2x^4} = 2$. Therefore $f(x,y) \to 2$ as $(x,y) \to 0$ along $y = x^2$.
Since different paths lead to different values, the given limit does not exist.

2. (8 pts.) Find the directional derivative of $f(x,y) = xy$ at the point $P(1,9)$ in the direction from P to $Q(4,5)$. Is $f(x,y)$ (circle one) increasing / decreasing / stationary at P?

Sol. Given $f(x,y) = xy$. Gradient of $f(x,y)$= $\nabla f(x,y) = \langle y, x\rangle$. $\overline{PQ} = \langle 4-1, 5-9\rangle = \langle 3, -4\rangle$.
The unit vector in the direction of $\overline{PQ} = \hat{u} = \dfrac{\langle 3,-4\rangle}{\sqrt{3^2+(-4)^2}} = \dfrac{1}{5}\langle 3,-4\rangle$.
The directional derivative of f in the direction of $\hat{u}$= $D_{\hat{u}}f(x,y) = \langle y,x\rangle \cdot \dfrac{1}{5}\langle 3,-4\rangle = \dfrac{1}{5}(3y-4x)$.
Thus $D_{\hat{u}}f(1,9) = \dfrac{1}{5}(3(9)-4(1)) = \dfrac{23}{5}$.
Increasing.

3. (8 pts.) Suppose that

$$f(x,y) = x\,e^{xy} \qquad \text{where } x = t^2,\ y = \ln(t).$$

Use the **Chain Rule** to find the derivative $\frac{df}{dt}$. Simplify your answer completely for full credit and make sure it is a function only of the variable t.

Sol. Given $f(x,y) = xy$. Then $\dfrac{\partial f}{\partial x} = xe^{xy}y + e^{xy} = (xy+1)e^{xy}$ and $\dfrac{\partial f}{\partial y} = x^2e^{xy}$.
Moreover, $x = t^2$ then $\dfrac{dx}{dt} = 2t$ and $y = \ln(t)$ then $\dfrac{dy}{dt} = \dfrac{1}{t}$.
Now,

$$\begin{aligned}
\frac{df}{dt} &= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} \\
&= (xy+1)e^{xy}2t + x^2e^{xy}\frac{1}{t} \\
&= (2t(xy+1) + x^2/t)e^{xy} \\
&= (2t^3\ln t + 2t + t^3)t^{t^2}.
\end{aligned}$$

4. (12 pts.) Consider the surface defined by $h(x, y) = 5x^2 + 3y^2$.

(a) Find the tangent plane to the surface $h(x, y) = 5x^2 + 3y^2$ at the point $(1, 1, h(1, 1))$.

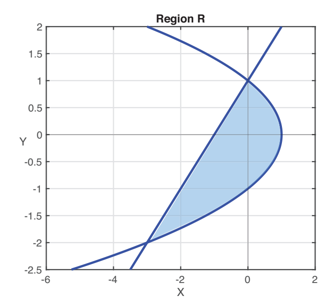
Sol. Given $h(x, y) = 5x^2 + 3y^2$. So, $h(1, 1) = 5 + 3 = 8$.
$h_x(x, y) = 10x, h_x(1, 1) = 10$ and $h_y(x, y) = 6y, h_y(1, 1) = 6$.
The equation of the tangent plane to the surface $h(x, y) = 5x^2+3y^2$ at the point $(1, 1, h(1, 1))$ is $z - 8 = 10(x - 1) + 6(y - 1)$.
Or $z = 10x + 6y - 8$.

(b) Estimate the value $h(.9, 1.01)$ using differentials. (Full credit only for using a linear approximation.)

Sol. The linear approximation of $h(x, y)$ at $(1, 1)$ is given by

$$\begin{aligned} h(x, y) &= h(1, 1) + h_x(1, 1)(x - 1) + h_y(1, 1)(y - 1) \\ &= 8 + 10(x - 1) + 6(y - 1) \\ h(0.9, 1.01) &= 8 + 10(0.9 - 1) + 6(1.01 - 1) \\ &= 8 - 1 + 0.06 \\ &= 7.06. \end{aligned}$$

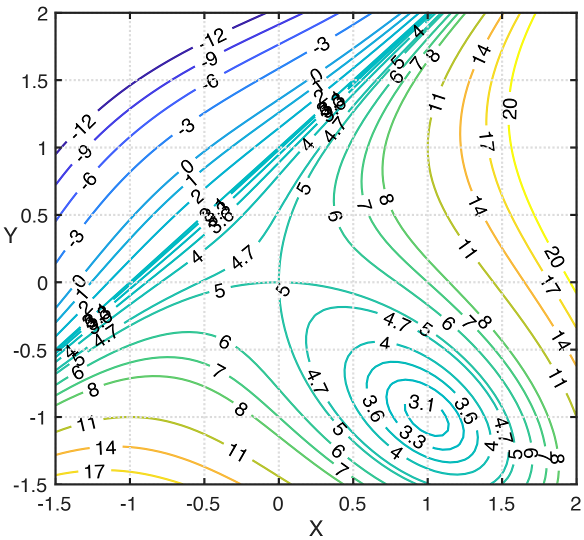
5. (12 pts.) The shaded lamina (plate or region) R below is bounded by the curves with equations $y^2 = 1-x$ and $y = x+1$. On this lamina, the charge density is given by $\sigma(x, y) = xy$ coulombs/m^2. Find the total charge of the lamina, including units in your final answer.



Sol. The total charge is given by $Q = \iint_D \sigma(x, y)dA = \iint_D xydA$.
Here $D = \{(x, y)| -2 \le y \le 1, y - 1 \le x \le 1 - y^2\}$. Then

$$\begin{aligned} Q &= \int_{-2}^{1}\int_{y-1}^{1-y^2} xy\ dxdy \\ &= \int_{-2}^{1} \frac{x^2}{2}y|_{y-1}^{1-y^2} dy \\ &= \frac{1}{2}\int_{-2}^{1} y(1 - y^2) - y(y - 1)^2 dy \\ &= \frac{1}{2}\int_{-2}^{1} (y^5 - 3y^3 + 2y^2)dy \\ &= \frac{1}{2}(\frac{y^6}{6} - \frac{3}{4}y^4 + \frac{2}{3}y^3)|_{-2}^{1} \\ &= \frac{27}{8} \text{ coulomb.} \end{aligned}$$

6. (14 pts.) Pictured is a contour plot for the function $f(x,y) = 5 + 2x^3 - 2y^3 + 6xy$



(a) The function $f(x,y)$ has **two** local extrema at points (a,b), [i.e. a saddle point, a local maximum, or a local minimum at (a,b)]. In the table below, give the values of these extrema and the points at which they occur. Then briefly justify your answer.

	coordinates (a,b)	Value $f(a,b)$	min, max or saddle ?
1.	(0, 0)	f(0,0)=5	saddle point
2.	(1, -1)	f(1, -1)=3.1	local min

Justification: Critical point $(0,0)$ is the intersection of two contour lines. If we move toward $(0,0)$ along the line $y = x$, f decreases. But if we move toward $(0,0)$ along the line $y = -x$, f increases. Thus $(0,0)$ is a saddle point.
Critical point $(1,-1)$ is the center of all the contour lines and all contours decreases as we move toward $(1,-1)$. Thus it is a local min.

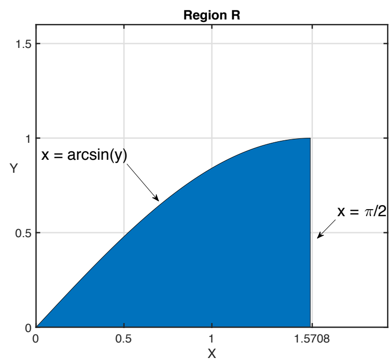
(b) Use the second derivatives test to verify your answer. That is, find all critical points of $f(x,y)$ and classify them as local maxima, local minima, or saddle points.

Sol. Given $f(x,y) = 5 + 2x^3 - 2y^3 + 6xy$, then $f_x(x,y) = 6x^2 + 6y$, $f_y(x,y) = -6y^2 + 6x$, $f_{xx}(x,y) = 12x$, $f_{xy}(x,y) = 6$, and $f_{yy}(x,y) = -12y$.
Solving $f_x(x,y) = 0$ and $f_y(x,y) = 0$ find the critical points $(0,0)$ and $(1,-1)$.
$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}^2(x,y) = -144xy - 36$
$D(0,0) = -36 < 0$ and $D(1,-1) = 144 - 26 = 108 > 0$.

	Critical Point (a,b)	Value $f(a,b)$	f_{xx}	D	min, max or saddle ?
1.	(0, 0)	f(0,0)=5	0	-36	saddle point
2.	(1, -1)	f(1, -1)=3	12	108	local min

7. (12 pts.) Compute the double integral over the region R of integration by **reversing the order of integration.**

$$\int_0^1 \int_{\arcsin(y)}^{\frac{\pi}{2}} \cos(x)\sqrt{3+\cos^2(x)}\,dx\,dy$$



If we reverse the order then

$$\int_0^1 \int_{x=\arcsin(y)}^{\frac{\pi}{2}} \cos(x)\sqrt{3+\cos^2(x)}\,dx\,dy = \int_0^{\frac{\pi}{2}} \int_{y=0}^{sinx} \cos(x)\sqrt{3+\cos^2(x)}\,dy\,dx.$$

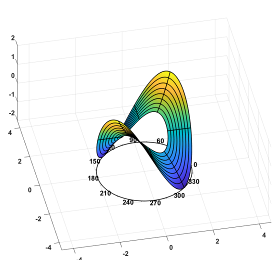
Then

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \int_{y=0}^{sinx} \cos(x)\sqrt{3+\cos^2(x)}\,dy\,dx \\ = &\int_0^{\frac{\pi}{2}} \cos(x)\sin(x)\sqrt{3+\cos^2(x)}\ dx \end{aligned}$$

Let $3+\cos^2(x) = u$ then $-2\cos(x)\sin(x)dx = du$, so

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \cos(x)\sin(x)\sqrt{3+\cos^2(x)}\ dx \\ = &\frac{1}{2}\int_3^4 \sqrt{u}du \\ = &\frac{1}{2}\frac{u^{3/2}}{3/2}\bigg|_3^4 = \frac{1}{3}(8-3\sqrt{3}). \end{aligned}$$

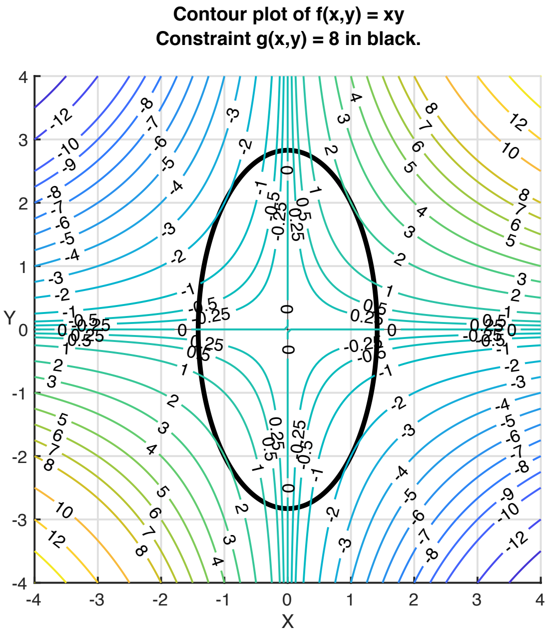
8. (12 pts.) Find the surface area of the part of the saddle $z = x^2 - y^2$ that lies between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$. (A picture is included for help with visualization, but is unnecessary.)



Given $z = x^2 - y^2$, so $\frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = -2y$. The surface area is $A = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$. Where D is the region between the cylinders $x^2+y^2=1$ and $x^2+y^2=2^2$. In polar coordinates: $D = \{(r,\theta)|1 \le r \le 2, 0 \le \theta \le 2\pi\}$.

$$\begin{aligned} A &= \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} \\ &= \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2}\ rdrd\theta \\ &= \int_0^{2\pi} d\theta \int_1^2 \sqrt{1+4r^2}\ rdr \\ &= \frac{1}{8}2\pi\frac{(1+4r^2)^{\frac{3}{2}}}{\frac{3}{2}}\bigg|_1^2 = \frac{\pi}{6}(17\sqrt{17}-5\sqrt{5}). \end{aligned}$$

9. (14 pts.) Consider the function $f(x, y) = xy$ and its contour plot shown below.



(a) The function $f(x, y)$ has two local minima subject to the constraint $g(x, y) = 4x^2 + y^2 = 8$. (The constraint $g(x, y) = 8$ is plotted in black in the figure.) By examining the contour plot give the coordinates of the two local minima (a, b) and the value $f(a, b)$ at those points.

	(a, b)	Minimum value $f(a, b)$
1.	$(a_1, b_1) = (1, 2)$	$f(1, 2) = 2$
2.	$(a_2, b_2) = (-1, -2)$	$f(1, 2) = 2.$

(b) Give the equations you must solve simultaneously in order to use the method of Lagrange multipliers to find the minimum values of $f(x, y)$ subject to the constraint $4x^2 + y^2 = 8$. (Be careful; it might be easy to leave out one equation.)

Sol. The gradient vectors are $\nabla f(x, y) = \langle y, x\rangle$ and $\nabla g(x, y) = \langle 8x, 2y\rangle$. Thus the equations we need to solve are

$$\begin{aligned} y &= \lambda 8x \\ x &= \lambda 2y \\ 4x^2 + y^2 &= 8. \end{aligned}$$

(c) Now verify that the first point, call its coordinates (a_1, b_1), in your list from part (a) satisfies these equations.
Sol. From part (b) using first two equations we get $\lambda = \pm\frac{1}{4}$. For $\lambda = \frac{1}{4}$ and the first point $(1, 2)$ we have $2 = \frac{1}{4}(8)(1) = 2$, $1 = \frac{1}{4}(2)(2) = 1$ and $4(1)^2 + 2^2 = 8$.

(d) One of the equations you gave in (b) should involve the gradient vector ∇f. Compute the gradient vectors $\nabla f(a_1, b_1)$ and $\nabla g(a_1, b_1)$, then plot them (up to a positive scaling factor) in the contour plot above. Then in the space to the right, explain briefly why the method of Lagrange multipliers works.

$\nabla f(a_1, b_1) = \langle 2, 1\rangle$

$\nabla g(a_1, b_1) = \langle 8, 4\rangle$

Explanation: The positive factor is $\frac{1}{4}$. We observe that the vector $\langle 2, 1\rangle$ at $(1, 2)$ are normal to both the contours $g(x, y) = 8$ and $f(x, y) = 2$. Thus the Lagrange method works here.