MATH253X-UX1
Spring 2016

Midterm Exam 2

Name: Answer Key

Instructions. You have 60 minutes. Closed book, closed notes, no calculator. *Show all your work* in order to receive full credit.

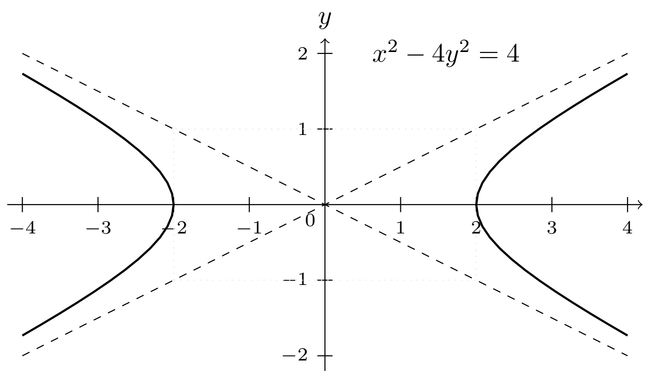
1. Show that $\displaystyle\lim_{(x,y)\to(2,-1)} \frac{xy+2}{x^2+4y}$ does not exist.

Solution: Setting $x=2$ and letting $y\to -1$ to approach $(2,-1)$ along the line $(2,y)$, we see $\displaystyle\lim_{y\to -1}\frac{2y+2}{4+4y} = \frac{1}{2}$. Setting $y=-1$ and letting $x\to 2$ to approach $(2,-1)$ along the line $(x,-1)$, we see $\displaystyle\lim_{x\to 2}\frac{2-x}{x^2-4} = -\frac{1}{4}$. Since these limits are different, $\boxed{\text{the original multivariable limit does not exist.}}$

2. Consider the function $z=f(x,y)=x^2-4y^2$.

(a) Sketch the level curve $z=4$.

Solution:



(b) Use Lagrange multipliers to find the absolute maximum $z_{\max}$ of f on the line $2x+y=15$.

Solution: Define $g(x,y)=2x+y$. Then we must solve (along with the constraint $g(x,y)=15$):

$$\nabla f=\lambda\nabla g \iff \langle 2x,-8y\rangle=\lambda\langle 2,1\rangle \iff \begin{cases}2x=2\lambda\\ -8y=\lambda\end{cases} \iff \lambda=x=-8y$$

Substituting into the constraint, we have:

$$2(-8y)+y=15 \iff y=-1$$

and so $x=-8(-1)=8$. Thus,

$$z_{\max}=f(8,-1)=64-4=\boxed{60}.$$

(c) What is the geometrical relationship between $2x+y=15$ and the level curves $z=z_{\max}$ at their intersection?

Solution: Note that the line $2x+y=15$ can be parametrized by $\langle x,15-2x\rangle$ so its direction is $\langle 1,-2\rangle$ and $\nabla g\perp\langle 1,-2\rangle$. But then ∇f is orthogonal to (the tangent of) the level curve at any point so also at $(8,-1)$ along $z=z_{\max}$. Now since ∇f and ∇g are parallel, then so are the line and the tangent to the level curve. They also share the point $(8,-1)$ so

$\boxed{\text{the line } 2x+y=15 \text{ is the tangent to the level curve } z=z_{\max} \text{ at } (8,-1)}$.

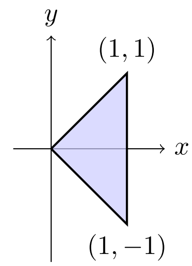
3. Consider the double integral:

$$I = \iint_R e^{x^2}\, dA$$

where R is the triangular region with vertices $(0,0)$, $(1,1)$, and $(1,-1)$.

(a) Write I as an iterated integral in two ways.

Solution: Let's sketch the region R:



We have the lines $y = x$, $y = -x$ and $x = 1$ so the double integral can be written as either of these forms in rectangular coordinates:

$$\boxed{I = \int_0^1 \int_{-x}^{x} e^{x^2}\, dy\, dx = \int_{-1}^{0} \int_{-y}^{1} e^{x^2}\, dx\, dy + \int_0^1 \int_y^1 e^{x^2}\, dx\, dy}$$

(b) Compute the integral using the form of your choice.

Solution: Note that we need to use the $dy\,dx$ order because e^{x^2} can not be integrated directly wrt x using elementary functions:

$$\begin{aligned} I &= \int_0^1 \int_{-x}^{x} e^{x^2}\, dy\, dx = \int_0^1 \left[y e^{x^2} \right]_{y=-x}^{y=x} dx \\ &= \int_0^1 2x e^{x^2}\, dx = \left| \begin{matrix} u = x^2 & du = 2xdx \\ x = 0 & u = 0 \\ x = 1 & u = 1 \end{matrix} \right| \\ &= \int_0^1 e^u\, du = \left[e^u \right]_0^1 = \boxed{e - 1} \end{aligned}$$

4. Find an equation of the tangent plane to the surface

$$x^2 y - z^2 + \ln(x+y) = 1$$

at the point $(x_0, y_0, z_0) = (-1, 2, 1)$.

Solution: For $F(x,y,z) = x^2 y - z^2 + \ln(x+y) = 1$, we find

$$\nabla F(x,y,z) = \left\langle 2xy + \frac{1}{x+y}, x^2 + \frac{1}{x+y}, -2z \right\rangle,$$

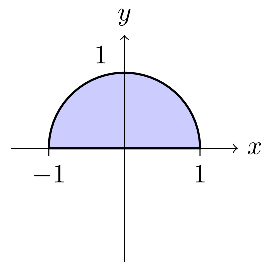
so $\nabla F(-1,2,1) = \langle -3, 2, -2 \rangle$. The tangent plane is thus given by

$$-3(x+1) + 2(y-2) - 2(z-1) = 0,$$

or

$$\boxed{-3x + 2y - 2z = 5.}$$

5. Compute the mass m of the planar lamina with density $\rho(x,y) = x^2y$ shown below.



Solution: Let's use polar coordinates: $\rho(r\cos\theta, r\sin\theta) = r^3\cos^2\theta\sin\theta$ and R will have constant bounds in (r,θ), that is $0 \le r \le 1$ and $0 \le \theta \le \pi$. Hence

$$m = \int_0^1 \int_0^\pi r^3\cos^2\theta\sin\theta\, r\, d\theta\, dr = \int_0^1 r^4\left[-\frac{\cos^3\theta}{3}\right]_0^\pi dr = \int_0^1 r^4\left(\frac{1}{3}+\frac{1}{3}\right) dr = \frac{2}{3}\left[\frac{r^5}{5}\right]_0^1 = \boxed{\frac{2}{15}}.$$

6. Find and classify all critical points of

$$f(x,y) = x^2y - 2x + 4y^2.$$

Solution: The gradient is

$$\nabla f = \langle f_x, f_y\rangle = \langle 2xy-2, x^2+8y\rangle$$

is defined everywhere and when setting it to the zero vector, we get $f_x = 0$ if $xy = 1$; that means $x, y \ne 0$ and $y = \frac{1}{x}$ so $f_y = 0$ becomes

$$x^2 + \frac{8}{x} = 0 \quad\Rightarrow\quad x^3 + 8 = 0 \quad\Rightarrow\quad x = -2$$

This in turns means $y = -\frac{1}{2}$. So we have one critical point $\boxed{\left(-2, -\frac{1}{2}\right)}$. To classify it, we use the Second Partials Test:

$$f_{xx} = 2y \quad , \quad f_{yy} = 8 \quad , \quad f_{xy} = 2x \quad\Rightarrow\quad d(x,y) = 16y - 4x^2$$

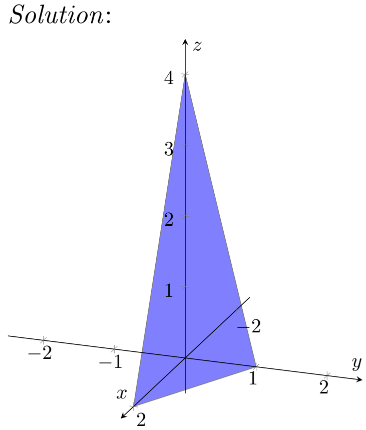
Now,

$$d\left(-2,-\frac{1}{2}\right) = -8 - 16 < 0 \text{ so } \boxed{\text{saddle point at } \left(-2,-\frac{1}{2},3\right)}.$$

7. Fully SET UP bounds and integrands but DO NOT EVALUATE the following double integrals.

(a) the volume below the plane $2x + 4y + z = 4$ in the first octant:

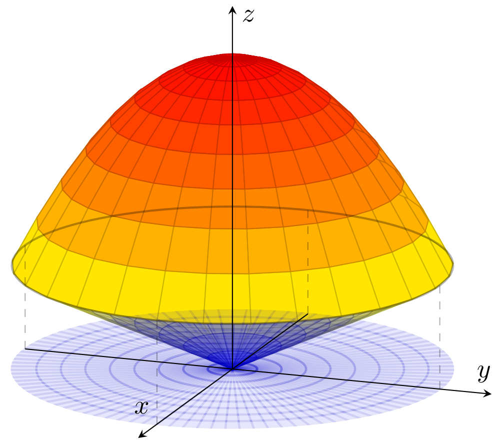
Solution:



Solve for $z = 4 - 2x - 4y$ for the integrand, then set $z = 0$ to get a boundary line in the xy-plane, the others being $x = 0$, $y = 0$. Finally for the order $dy\, dx$ set $y = 0$ in the line to get the upper constant bound in x. Thus we have:

$$\boxed{V = \int_0^2 \int_0^{1-\frac{x}{2}} 4 - 2x - 4y\, dy\, dx}$$

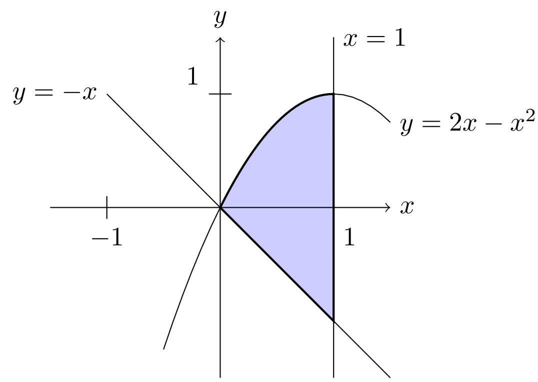
(b) the volume of the solid bounded by the cone $z = \sqrt{x^2+y^2}$ and the inverted paraboloid $z = 6-x^2-y^2$ using polar coordinates.



Solution: The cone is below the paraboloid and for the base, we have a disk where the radius can be found using the intersection of the surfaces, i.e. set $\sqrt{x^2+y^2} = 6-x^2+y^2$ or in polar $r = 6 - r^2$ for $r = \sqrt{x^2+y^2} \geq 0$. So $r^2 + r - 6 = 0$ which has for solutions $r = -3, 2$ and we keep $r = 2$. And so the volume is:

$$\boxed{V = \int_0^2 \int_0^{2\pi} (6 - r^2 - r)\, r \, d\theta \, dr}$$

(c) the surface area of $z = 4 - x^2 - y$ above the region R bounded by the graphs of $y = -x$, $y = 2x - x^2$, $x = 0$ and $x = 1$ as sketched below:



Solution:

The gradient is $\nabla z = \langle z_x, z_y \rangle = \langle -2x, -1 \rangle$ so noting that R is vertically simple, we have that the surface area of our surface above R is:

$$\boxed{SA = \int_0^1 \int_{-x}^{2x-x^2} \sqrt{4x^2+2}\, dy \, dx}$$

8. Let

$$f(x,y) = \frac{x}{x-y}.$$

(a) Compute the maximum rate of change of f at the point $(1,2)$ and specify a unit vector in the direction where this maximum change occurs.

Solution: The gradient is

$$\nabla f(x,y) = \left\langle \frac{1(x-y) - x(1)}{(x-y)^2}, \frac{x}{(x-y)^2} \right\rangle = \left\langle \frac{-y}{(x-y)^2}, \frac{x}{(x-y)^2} \right\rangle.$$

So the maximum rate of change of f at $(1,2)$ is:

$$\|\nabla f(1,2)\| = \|\langle -2, 1 \rangle\| = \boxed{\sqrt{5}},$$

and a unit direction of greatest increase is

$$\boxed{\frac{\nabla f(1,2)}{\|\nabla f(1,2)\|} = \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle.}$$

(b) Find the directional derivative of f at $(1,2)$ in the direction of $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$.

Solution: The direction we consider is $\mathbf{u} = \dfrac{\mathbf{v}}{||\mathbf{v}||} = \langle 2/\sqrt{13}, 3/\sqrt{13}\rangle$. Then

$$D_{\mathbf{u}}f(1,2) = \nabla f(1,2)\cdot\mathbf{u} = \langle -2, 1\rangle \cdot \langle 2/\sqrt{13}, 3/\sqrt{13}\rangle = \boxed{-1/\sqrt{13}.}$$

(c) Use the differential df to find an approximation of $f(1.1, 1.95)$.

Solution:

$$\begin{aligned} f(1.1, 1.95) &\approx f(1,2) + df \\ &= f(1,2) + f_x(1,2)(1.1-1) + f_y(1,2)(1.95-2) \\ &= -1 - 2(0.1) + 1(-0.05) = \boxed{-1.25} \end{aligned}$$