# Math F253 Midterm 2 Fall 2022

**Name:** Solutions

**Student Id:** ____________________

**Calculator Model:** ____________________

## Rules:

You have 70 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

A scientific or graphing calculator is allowed.

A one page sheet of paper (8 1/2 in. x 11 in.) with handwritten notes on one side is allowed.

No other aids are permitted.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

If you need extra space, you can use the back sides of the pages. Please make it obvious when you have done so.

Turn off anything that might go beep during the exam.

Good luck!

| Problem | Possible | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| Extra Credit | 5 | |
| Total | 60 | |

**1. (10 points)**

A quantity of a little more than a mole of gas satisfies

$$PV = 9T$$

where pressure $P$ is measured in kPa, volume $V$ is in liters and temperature $T$ is in Kelvin.

**a.** At time $t = 0$ the temperature of the gas is 300 Kelvin and the container containing it has a volume of 30 liters. What is the pressure of the gas, including units?

$$P = \frac{9T}{V} \Rightarrow P = \frac{9 \cdot 300}{30} = \boxed{90 \text{ kPa}}$$

**b.** At time $t = 0$ we know additionally that the temperature of the gas is rising at a rate of $dT/dt = 10$ K/hour and that the volume of the container is decreasing at a rate $dV/dt = -2$ liters/hour. Determine $dP/dt$, including all units.

$$\frac{dP}{dt} = \frac{\partial P}{\partial V} \cdot \frac{dV}{dt} + \frac{\partial P}{\partial T} \cdot \frac{dT}{dt}$$

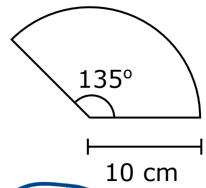
$$= -\frac{9T}{V^2} \cdot \frac{dV}{dt} + \frac{9}{V} \frac{dT}{dt}$$

$$= -\frac{9 \cdot 300}{(30)^2} \cdot (-2) + \frac{9}{30} \cdot 10$$

$$= 6 + 3 = \boxed{9 \text{ kPa/hour}}$$

**2. (10 points)**

A thin metal plate has the shape below:




Misprint on original exam $5\cdot(r-10)$ generates a sign error.

The mass density of the plate is given by $\rho = 5(10-r)$ grams per square centimeter, where $r$ is the distance in centimeters from the vertex of the plate.

Compute the mass of the plate. **Hint:** the equation we have for area in polar coordinates was for angles in radians.

$$135^\circ = \frac{3\pi}{4} \text{ rad}$$

$$\int_0^{3\pi/4}\int_0^{10} 5\cdot(10-r)\, r\, dr\, d\theta$$

$$= \int_0^{3\pi/4}\int_0^{10} 50r - 5r^2\, dr\, d\theta$$

$$= \int_0^{3\pi/4} \left.\frac{50r^2}{2} - \frac{5r^3}{3}\right|_{r=0}^{10} d\theta$$

$$= \int_0^{3\pi/4} \frac{5000}{6}\, d\theta$$

$$= \frac{3\pi}{4}\cdot\frac{5000}{6} = \boxed{625\pi \text{ g}}$$

**3. (10 points)**

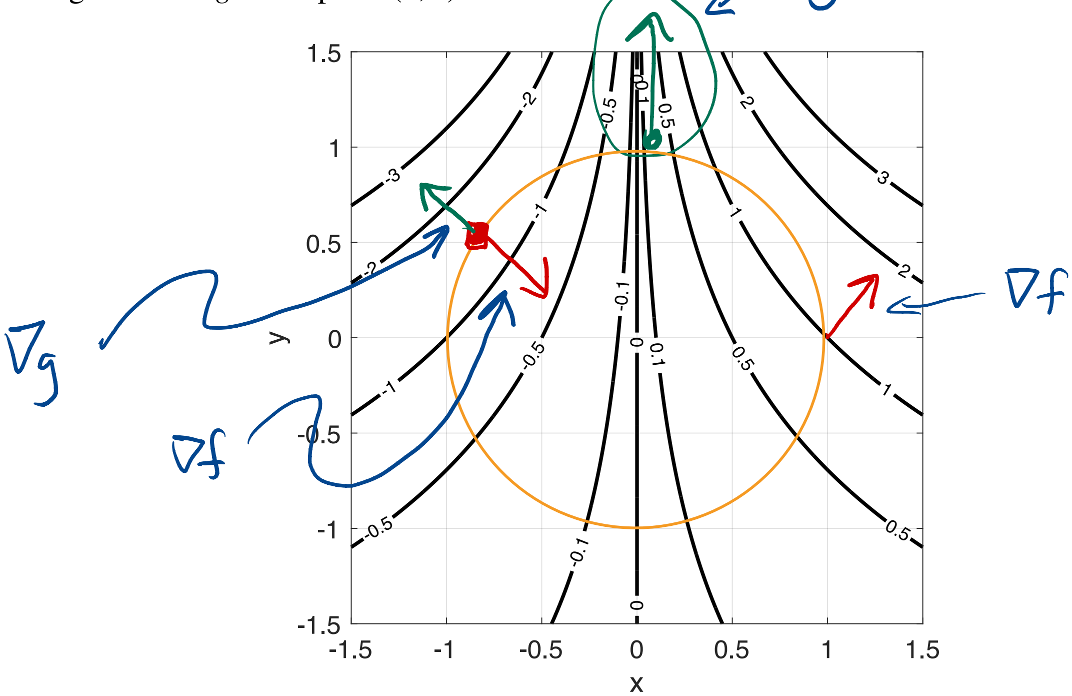
We wish to optimize the function

$$f(x, y) = xe^y$$

subject to the constraint $g(x, y) = x^2 + y^2 = 1$.

**a.** The figure below depicts level sets of $f$. Add the following to the figure:

- The gradient of $f$ at the point $(1, 0)$
- The constraint set (i.e. all the points satisfying $g(x, y) = 1$)
- The gradient of $g$ at the point $(0, 1)$




**b.** In the figure, label **with a square** the location on the constraint set where $f$ is **minimized**. At that point also add to the figure the gradient of $f$ and the gradient of $g$. Clearly label which gradient goes with which function.

**c.** Set up the system of equations to solve (using the method of Lagrange multipliers) to determine the location of the minimizer. **Do not solve the system!!**

$\vec{\nabla} f = \lambda \vec{\nabla} g$, $g(x,y) = 1$

$\vec{\nabla} f = \langle e^y, xe^y \rangle$, $\vec{\nabla} g = \langle 2x, 2y \rangle$

$$\boxed{\begin{aligned} e^y &= 2\lambda x \\ xe^y &= 2\lambda y \\ x^2 + y^2 &= 1 \end{aligned}}$$

**4. (10 points)**

Determine all critical points of the function

$$f(x, y) = y(e^x - 1)$$

and classify each as a local min, local max or saddle.

$$\vec{\nabla} f = \langle ye^x, e^x - 1 \rangle$$

$$\vec{\nabla} f = \vec{0} \Rightarrow \quad ye^x = 0 \Rightarrow y = 0$$

$$e^x - 1 = 0 \Rightarrow x = 0$$

$$\boxed{\text{One critical point: } (0,0)}$$

Hessian: $f_{xx} = ye^x \quad f_{xy} = e^x \quad f_{yy} = 0$

$$\begin{bmatrix} ye^x & e^x \\ e^x & 0 \end{bmatrix}$$

Discriminant: $D = ye^x \cdot 0 - e^{2x} = -e^{2x}$

At $(0,0)$ $D = -e^0 = -1 < 0$

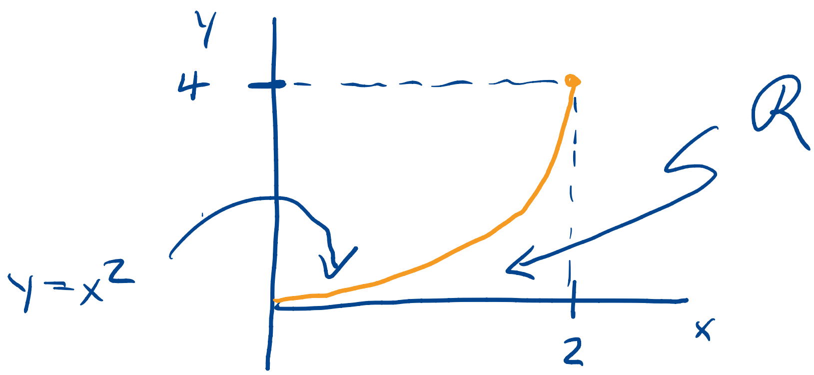
$$\boxed{\Rightarrow \text{saddle}}$$

**5. (10 points)**

Consider the following iterated integral:

$$\int_0^4 \int_{\sqrt{y}}^2 \cos x^3 dxdy.$$

**a.** This iterated integral computes a double integral over a region $\mathcal{R}$. Carefully sketch the region $\mathcal{R}$.




**b.** Interchange the order of integration to compute the integral.

$$\int_0^2 \int_0^{x^2} \cos(x^3)\,dy\,dx = \int_0^2 x^2 \cos(x^3)\,dx \qquad u = x^3 \quad du = 3x^2 dx$$

$$= \int_0^8 \cos(u)\,\frac{1}{3}\,du$$

$$= \left.\frac{\sin(u)}{3}\right|_0^8$$

$$= \boxed{\frac{\sin(8)}{3}}$$

**6. (10 points)**

Consider the function

$$F(x, y, z) = xy + yz + zx.$$

**a.** The point $(1, 2, 2)$ is on a level set of $F$. Name two other points that are on the same level set.

$F(1,2,2) = 8$
$F(2,1,2) = 8$
$F(1,2,2) = 8$

$\boxed{\begin{matrix}(2,1,2)\\(1,2,2)\end{matrix}}$

**b.** Compute the gradient of $F$ at the point $(1, 2, 2)$.

$\vec{\nabla} F = \langle y+z,\ x+z,\ y+x \rangle$

at $(1,2,2)$

$\boxed{\vec{\nabla} F = \langle 4, 3, 3 \rangle}$

**c.** Recall that the gradient of a function is perpendicular to the level sets. So the gradient just computed is a normal vector for the tangent plane to the level set of $F$ at the point $(1, 2, 2)$. Determine the equation of this tangent plane.

Point: $(1,2,2)$ Normal: $\langle 4,3,3 \rangle$

$\boxed{4(x-1) + 3(y-2) + 3(z-2) = 0}$

**7. (Extra Credit: 5 points)**

On the back side of this page, solve the Lagrange multiplier problem from question 3. For full credit you must determine both the location of the minimizer and the minimum value.

 See next page

System to solve:

$$e^y = 2\lambda x$$
$$x e^y = 2\lambda y$$
$$x^2+y^2 = 1$$

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$e^y = 2\lambda x$ and $x e^y = 2\lambda y$

$$\Rightarrow \quad x \cdot (2\lambda x) = 2\lambda y$$
$$\Rightarrow \quad \lambda x^2 = \lambda y$$

But $e^y = 2\lambda x \Rightarrow \lambda \neq 0$ (as $e^y \neq 0$), so

we can divide by $\lambda$:

$$x^2 = y.$$

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Constraint: $x^2+y^2 = 1 \Rightarrow y + y^2 = 1$

$$\Rightarrow y^2+y-1 = 0$$
$$\Rightarrow y = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

But $y = x^2 \geq 0$ so only one choice is viable:

$$y = \frac{-1+\sqrt{5}}{2}.$$

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Since $y = x^2$, $x = \pm\sqrt{y} \Rightarrow x = \pm\sqrt{\frac{\sqrt{5}-1}{2}}$

Two solutions: $\left(-\sqrt{\frac{\sqrt{5}-1}{2}}, \frac{\sqrt{5}-1}{2}\right), \left(+\sqrt{\frac{\sqrt{5}-1}{2}}, \frac{\sqrt{5}-1}{2}\right)$

Max and min occur only at ~~these~~ these points.

If $x<0$ $f(x,y)=xe^y<0$. If $x>0$ $f(x,y)>0$.

So min value is $-\sqrt{\frac{\sqrt{5}-1}{2}}\, e^{(\sqrt{5}-1)/2}$

at $\left(-\sqrt{\frac{\sqrt{5}-1}{2}}, \frac{\sqrt{5}-1}{2}\right)$