MATH 253 Name: SOLUTIONS

Midterm 1 February 4, 2021

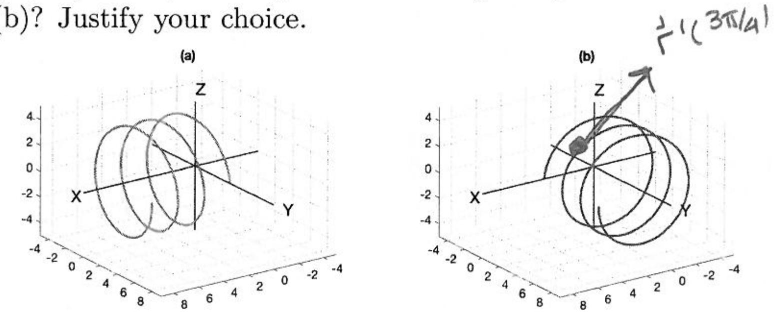
Instructions: 100 points total. Use only your brain and writing implement. You have 70 minutes to complete this exam. Good luck.

1. (26 pts.) The trajectory of a particle in $\mathbb{R}^3$ is given by the equation

$$\mathbf{r}(t) = \langle 4\cos(3t),\, t,\, 4\sin(3t)\rangle$$

where t is measured in seconds and $\mathbf{r}(t)$ in meters.

(a) (5 pts.) Two possible plots of this trajectory are drawn. Is the correct plot given by (a) or (b)? Justify your choice.



Answer: (b) Momentarily ignoring $y(t)$, we see $x(t), z(t)$ give a circle of radius 4. They $y(t)$ turns it into a helix along the y-axis.

(b) (7 pts.) Give the tangent vector to the curve at the time $t = \frac{3\pi}{4}$. Then on the (correct) plot above, sketch and label the location of the particle (as a point) and the tangent vector at the time $t = \frac{3\pi}{4}$. (Do not worry about getting the magnitude correct.)

$\vec{r}(t) = \langle 4\cos(3t),\, t,\, 4\sin(3t)\rangle \Rightarrow \vec{r}\,'(t) = \langle -12\sin(3t),\, 1,\, 12\cos(3t)\rangle$

and $\vec{r}\,'(\frac{3\pi}{4}) = \langle -12\sin(\frac{9\pi}{4}),\, 1,\, 12\cos(\frac{9\pi}{4})\rangle = \langle -12\sin(\frac{\pi}{4}),\, 1,\, 12\cos(\frac{\pi}{4})\rangle$

$= \boxed{\langle -6\sqrt{2},\, 1,\, 6\sqrt{2}\rangle \text{ tangent vector}}$

Position $\vec{r}(\frac{3\pi}{4}) = \langle 4\cos(\frac{9\pi}{4}),\, \frac{3\pi}{4},\, 4\sin(\frac{9\pi}{4})\rangle = \boxed{\langle 2\sqrt{2},\, \frac{3\pi}{4},\, 2\sqrt{2}\rangle}$

(c) (6 pts.) Give the speed of the particle at the time $t = \frac{3\pi}{4}$, including units in your final answer.

Speed $= |\vec{r}\,'(t)| = \sqrt{(-6\sqrt{2})^2 + 1^2 + (6\sqrt{2})^2} = \sqrt{72 + 1 + 72} = \boxed{\sqrt{145}\ \text{m/s}}$

(continued on next page.)

(d) (8 pts.) Find the distance traveled by the particle between the time $0 \le t \le \frac{\pi}{3}$, including units in your final answer.

Distance = Arc length

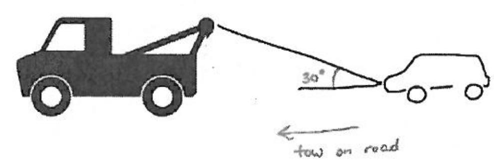
$$= \int_0^{\pi/3} |\vec{r}\,'(t)|\,dt = \int_0^{\pi/3} \sqrt{(-12\sin(3t))^2 + 1^2 + (12\cos(3t))^2}\,dt$$

$$= \int_0^{\pi/3} \sqrt{144\sin^2(3t) + 144\cos^2(3t) + 1}\,dt$$

$$= \int_0^{\pi/3} \sqrt{145}\,dt$$

$$= \boxed{\sqrt{145}\,\frac{\pi}{3} \text{ meters}}$$

2. (10 pts.) Your car breaks down and is pulled for a distance of 5 kilometers by a tow truck on a flat road. See Figure. Determine the amount of work done in towing your car if a constant force of magnitude 300 Newtons on the tow chain is applied. Include units in your final answer.



$$W = \vec{F}\cdot\vec{d} = |\vec{F}||\vec{d}|\cos 30°$$

$$= 300 \cdot 5 \frac{\sqrt{3}}{2}$$

$$= \boxed{750\sqrt{3} \text{ N-km}} = 750{,}000\sqrt{3} \text{ Nm}$$

or joules

3. (10 pts.) A bolt is tightened by applying a 20-Newton force to a .25 meter wrench as shown.

(a) (7 pts.) Find the magnitude of the torque vector, $|\boldsymbol{\tau}|$, about the center of the bolt. Include units in your answer.

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}|\sin(\pi/3) = (.25)(20)\frac{\sqrt{3}}{2}$$

$$= \boxed{\frac{5\sqrt{3}}{2} \text{ N}\cdot\text{m}}$$ or joules

(b) (3 pts.) In what direction does the torque vector $\boldsymbol{\tau}$ point?

Using the right hand rule, $\vec{\tau}$ points into the page.

4. (22 pts.) Consider the two vectors in $\mathbb{R}^3$:

$$\mathbf{a} = \langle -1,\ 2,\ 5\rangle \quad \text{and} \quad \mathbf{b} = \langle 3,\ -2,\ 1\rangle$$

(a) (4 pts.) Determine if the angle between the two vectors is acute, right, or obtuse. Briefly justify your answer.

$\vec{a}\cdot\vec{b} = \langle -1, 2, 5\rangle \cdot \langle 3, -2, 1\rangle = -3-4+5 = -2 < 0$

Since the dot product is negative θ is $\boxed{\text{obtuse.}}$

(b) (5 pts) Give the vector projection of **b** onto **a** and find its length.

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{-2}{\left(\sqrt{(-1)^2+(2)^2+(5)^2}\right)^2}\langle -1,2,5\rangle$$

$$= \frac{-2}{30}\langle -1,2,5\rangle = \frac{-1}{15}\langle -1,2,5\rangle$$

$$\text{Length} = \text{Comp}_{\vec{a}}\vec{b} = \left|\tfrac{-1}{15}\right||\vec{a}| = \tfrac{+1}{15}\sqrt{30}$$

Answer: $\mathbf{proj_a}(\mathbf{b}) = \frac{-1}{15}\langle -1,2,5\rangle = \langle \frac{1}{15}, \frac{-2}{15}, \frac{-1}{3}\rangle$ and its length is $\frac{\sqrt{30}}{15}$

(c) (8 pts) Noting that $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$, find the equation of the plane that is parallel to the plane containing the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ *AND* passes through the point $P(2, 0, -1)$.

To find a normal vector, take $\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 5 \\ 3 & -2 & 1 \end{vmatrix} = (2+10)\hat{i} - (-1-15)\hat{j} + (2-6)\hat{k} = \langle 12, 16, -4\rangle.$

Might as well take $\vec{n} = \frac{1}{4}\langle 12, 16, -4\rangle = \langle 3, 4, -1\rangle$ for ease. Thus, $\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{a}$ yields

$\langle 3,4,-1\rangle \cdot \langle x,y,z\rangle = \langle 3,4,-1\rangle\cdot\langle 2,0,-1\rangle$

$3x+4y-z = \underbrace{6+0+1}_{7}$

$\boxed{3x+4y-z=7}$

(d) (5 pts) Find the area of the parallelogram spanned by the vectors **a** and **b**.

$$\text{Area} = |\vec{a}\times\vec{b}| = \sqrt{12^2+16^2+(-4)^2} = \sqrt{144+256+16} = \sqrt{16(9+16+1)}$$

$$= 4\sqrt{26}$$

$$(=\sqrt{416})$$

5. (20 pts) Consider the two planes given by equations:

$$\begin{array}{lcccccccc} \text{Plane 1:} & x & + & 2y & - & z & = & 3 \\ \text{Plane 2:} & 3x & - & y & + & z & = & 2 \end{array}$$

(a) (5 pts.) Carefully prove that the two planes are not parallel.

$\vec{n}_1 = \langle 1, 2, -1 \rangle$ $\quad \vec{n}_2 = \langle 3, -1, 1 \rangle$ $\quad$ Since $\vec{n}_1 \neq c\vec{n}_2$ for any scalar c, the normal vectors and therefore the planes are <u>NOT</u> parallel.

(b) (5 pts.) Find the angle θ between the two planes.

Find the angle between $\vec{n}_1$ and $\vec{n}_2$ using $\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\theta$

$\vec{n}_1 \cdot \vec{n}_2 = \langle 1, 2, -1 \rangle \cdot \langle 3, -1, 1 \rangle = 3 - 2 - 1 = 0$

$$\boxed{\theta = \pi/2 \text{ or } 90^\circ}$$

(c) (10 pts.) Find the equation of the line $\ell(t)$ of intersection of the two planes. (You may give your answer in vector or parametric form.)

A direction vector is indicated by $\vec{v} = \vec{n}_1 \times \vec{n}_2$ since $\vec{v}$ would be orthogonal to $\vec{n}_1$ and $\vec{n}_2$ and therefore lie on <u>both</u> planes and therefore on $l(t)$. A point P on $l(t)$ is $\langle 1, 1, 0 \rangle$ which I got by inspecting the two equations.

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 3 & -1 & 1 \end{vmatrix} = (2-1)\hat{i} - (1+3)\hat{j} + (-1-6)\hat{k} = \langle 1, -4, -7 \rangle$$

Thus, $l(t) = \vec{p} + t\vec{v} = \langle 1, 1, 0 \rangle + t\langle 1, -4, -7 \rangle \quad t \in \mathbb{R}$

$x(t) = t + 1 \quad y(t) = -4t + 1 \quad z(t) = -7t$

Answer: $\ell(t) =$ ____.

6. (12 pts.) An atom moves on an electrified plate with acceleration given by

$$\mathbf{a}(t) = \left\langle e^{\cos(t)} \sin(t) + 2,\ \left(t - \frac{\pi}{2}\right)^2 \right\rangle \ \text{ft/s}^2$$

and velocity at time $t = \frac{\pi}{2}$ is

$$\mathbf{v}\left(\frac{\pi}{2}\right) = \langle \pi + 3,\ -1 \rangle .$$

Give the velocity of this atom at all times t. Include units in your final answer.

I will address the vector constant of integration last.

$\vec{v}(t) = \int \vec{a}(t)\,dt = \left\langle \int e^{\cos t} \sin t + 2\,dt,\ \int (t - \frac{\pi}{2})^2\,dt \right\rangle$

$\vec{v}_x(t) = \int e^{\cos t} \sin t + 2\,dt$ $\quad u = \cos t \quad du = -\sin t\,dt \quad \sin t\,dt = -du$

$= -\int e^u\,du + 2t = -e^u + 2t = \underline{-e^{\cos t} + 2t}$

$\vec{v}_y(t) = \int (t - \pi/2)^2\,dt = \underline{\frac{1}{3}(t - \pi/2)^3}$

Thus, $\vec{v}(t) = \langle -e^{\cos t} + 2t + c_x,\ \frac{1}{3}(t - \pi/2)^3 + c_y \rangle$

$\vec{v}(\pi/2) = \langle \pi + 3, -1 \rangle = \langle -e^{\cos(\pi/2)} + 2(\frac{\pi}{2}) + c_x,\ \frac{1}{3}(0)^3 + c_y \rangle$

$= \langle -1 + \pi + c_x, c_y \rangle \Rightarrow c_x = 4 \quad c_y = -1$

$$\boxed{\vec{v}(t) = \langle -e^{\cos(t)} + 2t + 4,\ \frac{1}{3}(t - \pi/2)^3 - 1 \rangle \ \text{ft/s}}$$

Some multiplied out

FORMULAS

$\vec{v}_y(t) = \frac{1}{3}t^3 - \frac{\pi}{2}t^2 + \frac{\pi^2}{4}t - \frac{\pi^3}{24} - 1$

The formulas for curvature $\kappa(t)$ for a space curve are:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \qquad \kappa = \frac{d\mathbf{T}}{ds}$$