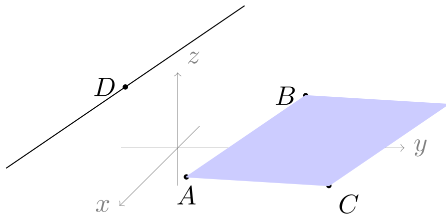
MATH253X-F01
Summer 2016

Midterm Exam 1

Name: Answer Key

Instructions. You have 90 minutes. No calculators allowed. *Show all your work* in order to receive full credit.

1. Consider the points $A(2,1,0)$, $B(-1,3,1)$, $C(0,4,-1)$, and $D(1,-1,2)$ in space.



(a) Find the symmetric equations of the line going through D and parallel to the line going through A and B.

Solution: The direction for the line will be (any scalar multiple of):

$$\overrightarrow{AB} = \langle -1-2, 3-1, 1-0\rangle = \langle -3, 2, 1\rangle$$

And symmetric equations for the line through D parallel to $\overrightarrow{AB}$ are:

$$\boxed{\frac{x-1}{-3} = \frac{y+1}{2} = z-2}$$

(b) Find the equation of the plane containing the parallelogram shaded above.

Solution: A normal vector to the plane is:

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= \langle -3,2,1\rangle \times \langle 0-2, 4-1, -1-0\rangle = \langle -3,2,1\rangle \times \langle -2,3,-1\rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & 1 \\ -2 & 3 & -1 \end{vmatrix} = \langle 2(-1)-3(1), -(-3(-1)+2(1)), -3(3)+2(2)\rangle = \langle -5,-5,-5\rangle\,.\end{aligned}$$

A more judicious choice is $\langle 1,1,1\rangle$ and then the equation to the plane is:

$$\boxed{(x-2)+(y-1)+z=0} \quad \text{or equivalently} \quad \boxed{x+y+z=3}.$$

(c) Use vectors to find the length of the diagonal starting at A in the parallelogram.

Solution: A vector representing the diagonal is $\overrightarrow{AB} + \overrightarrow{AC}$ so the length of the diagonal is:

$$\begin{aligned}\left\|\overrightarrow{AB} + \overrightarrow{AC}\right\| &= \|\langle -3,2,1\rangle + \langle -2,3,-1\rangle\| = \|\langle -3-2, 2+3, 1-1\rangle\| = \|\langle -5,5,0\rangle\| \\ &= \sqrt{25+25+0} = \sqrt{50} = \boxed{5\sqrt{2}}.\end{aligned}$$

2. Consider the following planes in space:

$$\begin{array}{ll} \text{Plane 1} & x - 2y - z + 1 = 0 \\ \text{Plane 2} & x - 3y + 2z + 6 = 0 \end{array}$$

(a) Are the two planes orthogonal?

Solution: We need to compute the dot product of the normal vectors:

$$\langle 1, -2, -1\rangle \cdot \langle 1, -3, 2\rangle = 1(1) - 2(-3) - 1(2) = 5 \neq 0$$

so $\boxed{\text{the planes are not orthogonal}}$.

(b) Find the point of intersection of Plane 1 and the line parametrized by

$$\vec{r}(t) = \langle -2 + t, 1 - t, 3 + 2t\rangle \,.$$

Solution: Substitute x, y, and z with the parametrization of the line in Plane 1 and solve for t:

$$(-2+t) - 2(1-t) - (3+2t) + 1 = 0 \quad \Longleftrightarrow \quad -2 + t - 2 + 2t - 3 - 2t + 1 = 0 \quad \Longleftrightarrow \quad t = 6.$$

Thus, we have

$$\vec{r}(6) = \langle -2+6, 1-6, 3+2(6)\rangle = \langle 4, -5, 15\rangle \quad \Longrightarrow \quad \boxed{P(4,-5,15)}.$$

(c) Now find the distance from the point found above to Plane 2.

Solution:

$$d = \frac{|4 - 3(-5) + 2(15) + 6|}{\|\langle 1, -3, 2\rangle\|} = \frac{|4 + 15 + 30 + 6|}{\sqrt{1+9+4}} = \frac{55}{\sqrt{14}} = \boxed{\frac{55\sqrt{14}}{14}}$$

3. Let $\mathbf{r}(t) = \left\langle (t-1)^2, t^3 - 3t^2 + 3t, 2t^3 - 3t^2 \right\rangle$ be describing the motion of a particle along a space curve over time. The position is in meters and time in seconds.

(a) Find all the open intervals on which the curve is smooth.

Solution: The domain of $\mathbf{r}(t)$ is the whole real line, and

$$\mathbf{r}'(t) = \left\langle 2(t-1), 3t^2 - 6t + 3, 6t^2 - 6t \right\rangle = \left\langle 2(t-1), 3(t-1)^2, 6t(t-1) \right\rangle$$

is defined, and continuous also on the whole real line. But

$$\mathbf{r}'(t) = \vec{0} \quad \text{for} \quad t = 1$$

So by definition,

$$\boxed{\text{the curve described by } \vec{r}(t) \text{ is smooth on } (-\infty, 1) \cup (1, \infty).}$$

(b) Find the speed of the particle at $t = 2$ s.

Solution:

$$\|\mathbf{r}'(2)\| = \left\| \left\langle 2(1), 3(1)^2, 6(2)(1) \right\rangle \right\| = \|\langle 2, 3, 12\rangle\| = \sqrt{4 + 9 + 144} = \boxed{\sqrt{157} \text{ m/s.}}$$

(c) Given $\mathbf{s}(2) = \langle 2, 3, -1\rangle$ and $\mathbf{s}'(2) = \langle 1, -1, 2\rangle$, find:

1. $\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s})\Big|_{t=2} = \boxed{8}$

Solution:

$$\begin{aligned}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s})\Big|_{t=2} &= (\mathbf{r}'\cdot\mathbf{s}+\mathbf{r}\cdot\mathbf{s}')\Big|_{t=2} = \langle 2,3,12\rangle\cdot\langle 2,3,-1\rangle + \left\langle 1^2, 8-12+6, 16-12\right\rangle\cdot\langle 1,-1,2\rangle \\ &= 2(2)+3(3)+12(-1)+\langle 1,2,4\rangle\cdot\langle 1,-1,2\rangle = 4+9-12+1(1)+2(-1)+4(2) \\ &= 1+1-2+8 = 8\end{aligned}$$

2. $\frac{d}{dt}(\mathbf{r}\times\mathbf{s})\Big|_{t=2} = \boxed{\langle -31, 28, -3\rangle}$

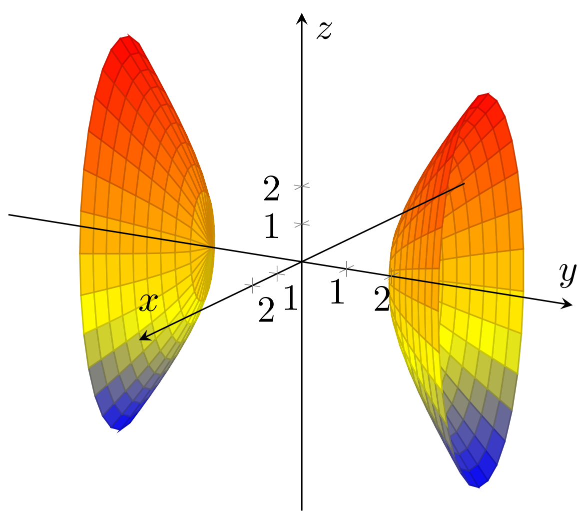
Solution:

$$\begin{aligned}\frac{d}{dt}(\mathbf{r}\times\mathbf{s})\Big|_{t=2} &= (\mathbf{r}'\times\mathbf{s}+\mathbf{r}\times\mathbf{s}')\Big|_{t=2} = \langle 2,3,12\rangle\times\langle 2,3,-1\rangle + \langle 1,2,4\rangle\times\langle 1,-1,2\rangle \\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 3 & 12\\ 2 & 3 & -1\end{vmatrix} + \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 4\\ 1 & -1 & 2\end{vmatrix} \\ &= \langle 3(-1)-3(12), -(2(-1)-2(12)), 2(3)-2(3)\rangle \\ &\qquad + \langle 2(2)+1(4), -(1(2)-1(4)), 1(-1)-1(2)\rangle \\ &= \langle -39, 26, 0\rangle + \langle 8, 2, -3\rangle = \langle -31, 28, -3\rangle\end{aligned}$$

4. Time to sketch some surfaces!

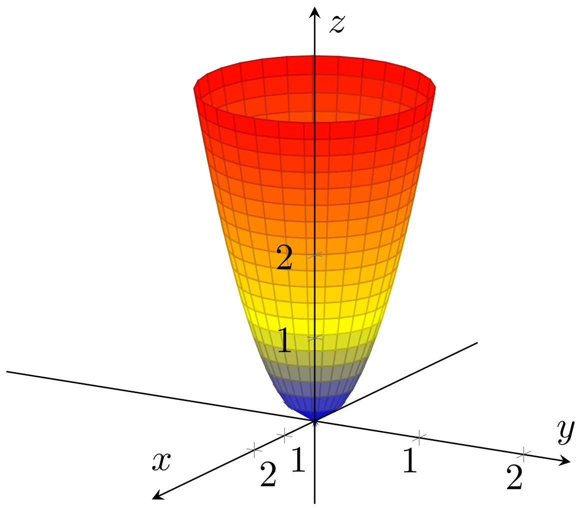
(a) Describe and sketch the surface in space whose equation is $x^2 - \frac{y^2}{4} + \frac{z^2}{9} = -1$.

Solution: hyperboloid of two sheets



(b) Describe and sketch the surface in space whose equation is: $x^2 + 4y^2 = z$.

Solution: elliptic paraboloid



5. You hit a golf ball in "Calculus III conditions"[1] such that it takes off at an angle of 30° with the horizontal. What should the initial golf ball speed be in order for you to hit a hole-in-one located 800 feet away at the same elevation?

Solution: Let v_0 be the original speed. Then the initial velocity is

$$\mathbf{v}(0) = \langle v_0 \cos 30°, v_0 \sin 30° \rangle = \left\langle \frac{v_0\sqrt{3}}{2}, \frac{v_0}{2} \right\rangle$$

and since the initial position is $\mathbf{r}(0) = \langle 0, 0 \rangle$, we have:

$$\mathbf{a}(t) = \langle 0, -32 \rangle \quad \Longrightarrow \quad \mathbf{v}(t) - \mathbf{v}(0) = \int_0^t \langle 0, -32 \rangle \, du = \langle 0, -32u \rangle \Big|_{u=0}^{u=t} = \langle 0, -32t \rangle$$

$$\Longleftrightarrow \quad \mathbf{v}(t) = \left\langle \frac{v_0\sqrt{3}}{2}, \frac{v_0}{2} \right\rangle + \langle 0, -32t \rangle = \left\langle \frac{v_0\sqrt{3}}{2}, \frac{v_0}{2} - 32t \right\rangle$$

$$\Longrightarrow \quad \mathbf{r}(t) - \mathbf{r}(0) = \int_0^t \left\langle \frac{v_0\sqrt{3}}{2}, \frac{v_0}{2} - 32u \right\rangle du = \left\langle \frac{v_0 u\sqrt{3}}{2}, \frac{v_0 u}{2} - 16u^2 \right\rangle \Bigg|_{u=0}^{u=t}$$

$$\Longrightarrow \quad \mathbf{r}(t) = \left\langle \frac{v_0 t\sqrt{3}}{2}, \frac{v_0 t}{2} - 16t^2 \right\rangle$$

Now we solve for t and v_0 in $\mathbf{r}(t) = \langle 800, 0 \rangle$. From the y-component,

$$\frac{v_0 t}{2} = 16t^2 \quad \Longleftrightarrow \quad t = 0 \quad \text{or} \quad t = \frac{v_0}{32}$$

and since $t = 0$ just gives $\mathbf{r}(0) = \langle 0, 0 \rangle$, here we need the second value of t and thus we solve from the x-component:

$$800 = \frac{v_0^2\sqrt{3}}{64} \quad \Longleftrightarrow \quad v_0 = \sqrt{\frac{800(64)}{\sqrt{3}}} = \boxed{\frac{160\sqrt{2}}{\sqrt[4]{3}} \text{ ft/s}}$$

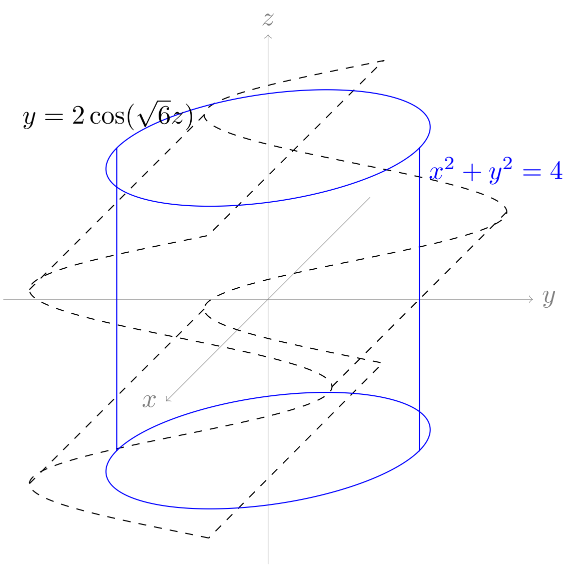
[1]I.e. the acceleration is constant and only due to gravity. That is we ignore ball spin, air resistance, etc.

6. An object moves along a trajectory so that its position $\vec{r}(t)$ as a function of time is given by:

$$\vec{r}(t) = \left\langle 2\sin(\sqrt{6}t), 2\cos(\sqrt{6}t), t \right\rangle.$$

(a) The particle's trajectory sits on the intersection of the cylinder $y = 2\cos(\sqrt{6}z)$ (drawn below) and which other surface? Sketch that surface. (*Hint:* Even though there is more than one possible answer, one is definitely easier to sketch.)

Solution:



$$x^2(t) + y^2(t) = 4\sin^2(\sqrt{6}t) + 4\cos^2(\sqrt{6}t) = 4$$

This is true for all t so the surface is the cylinder:

$$\boxed{x^2 + y^2 = 4}$$

(b) Find the unit tangent vector of the trajectory.

Solution:

$$\vec{r}'(t) = \left\langle 2\sqrt{6}\cos(\sqrt{6}t), -2\sqrt{6}\sin(\sqrt{6}t), 1 \right\rangle$$

$$\Longrightarrow \quad \vec{T}(t) = \frac{\vec{r}'(t)}{\left\|\vec{r}'(t)\right\|} = \frac{\langle 2\sqrt{6}\cos(\sqrt{6}t), -2\sqrt{6}\sin(\sqrt{6}t), 1\rangle}{\sqrt{24\cos^2(\sqrt{6}t) + 24\sin^2(\sqrt{6}t) + 1}} = \frac{\langle 2\sqrt{6}\cos(\sqrt{6}t), -2\sqrt{6}\sin(\sqrt{6}t), 1\rangle}{\sqrt{25}}$$

$$\Longrightarrow \quad \boxed{\vec{T}(t) = \left\langle \frac{2\sqrt{6}}{5}\cos(\sqrt{6}t), \frac{-2\sqrt{6}}{5}\sin(\sqrt{6}t), \frac{1}{5} \right\rangle}$$

(c) Find the principal unit normal vector of the trajectory.

Solution:

$$\vec{T}'(t) = \left\langle \frac{-12}{5}\sin(\sqrt{6}t), \frac{-12}{5}\cos(\sqrt{6}t), 0 \right\rangle = -\frac{12}{5}\left\langle \sin(\sqrt{6}t), \cos(\sqrt{6}t), 0 \right\rangle$$

$$\Longrightarrow \quad \vec{N}(t) = \frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|} = \frac{-\frac{12}{5}\langle \sin(\sqrt{6}t), \cos(\sqrt{6}t), 0\rangle}{\left|-\frac{12}{5}\right|\sqrt{\sin^2(\sqrt{6}t) + \cos^2(\sqrt{6}t) + 0}} = \frac{-\frac{12}{5}\langle \sin(\sqrt{6}t), \cos(\sqrt{6}t), 0\rangle}{\frac{12}{5}}$$

$$\Longrightarrow \quad \boxed{\vec{N}(t) = \left\langle -\sin(\sqrt{6}t), -\cos(\sqrt{6}t), 0 \right\rangle}$$

7. A particle is moving in the plane from a starting position at $(-1, 2)$ (i.e. $\vec{r}(0) = -\vec{\imath} + 2\vec{\jmath}$) according to the following ***velocity*** (measured in ft/s) at time t:

$$\vec{v}(t) = (3 - 3t^2)\vec{\imath} + 6t\vec{\jmath}.$$

(a) What is the particle's position at $t = 1$ s?

Solution:

$$\vec{r}(t) = \int \vec{v}(t)\,dt = (3t - t^3)\vec{\imath} + 3t^2\vec{\jmath} + \vec{c}$$

$$-\vec{\imath} + 2\vec{\jmath} = \vec{r}(0) = \vec{c} \quad \Longrightarrow \quad \vec{r}(t) = (3t - t^3 - 1)\vec{\imath} + (3t^2 + 2)\vec{\jmath} \quad \Longrightarrow \quad \vec{r}(1) = \vec{\imath} + 5\vec{\jmath}$$

so the position of the particle at $t = 1$ s is $\boxed{(1, 5)}$.

(b) Find the arc length described by the particle between $t = 0$ s and $t = 2$ s.

Solution: The distance traveled (or arc length) from $t = 0$ s to $t = 2$ s is:

$$\begin{aligned} s(2) &= \int_0^2 \|\vec{v}(t)\|\,dt = \int_0^2 3\sqrt{(1-t^2)^2 + (2t)^2}\,dt = 3\int_0^2 \sqrt{1 - 2t^2 + t^4 + 4t^2}\,dt \\ &= 3\int_0^2 \sqrt{1 + 2t^2 + t^4}\,dt = 3\int_0^2 \sqrt{(1+t^2)^2}\,dt = 3\int_0^2 1 + t^2\,dt \\ &= 3\left[t + \frac{t^3}{3}\right]_0^2 = 3\left(2 + \frac{8}{3} - 0\right) = \boxed{14 \text{ ft}} \end{aligned}$$

8. Nanook, your favorite sled dog is in training for pulling a loaded sled along a frictionless snow path.

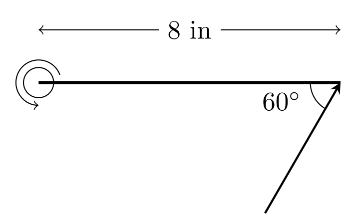
(a) He is applying a constant force $\vec{F}$ of magnitude 30 lbs along the rope which forms an angle of 10° with the horizontal path where the sled rests. What is the work done by this force when the sled is dragged over 20 ft? Your answer may still contain a trigonometric function but don't forget the overall unit.

Solution:

$$W = \vec{F} \cdot \overrightarrow{PQ} = \left\|\vec{F}\right\| \left\|\overrightarrow{PQ}\right\| \cos 10° = 30(20)\cos 10° = \boxed{600\cos 10° \text{ ft-lbs}}$$

(b) When Nanook reaches the dog yard, he runs to the cabin and pushes with his nose against the door with the same force, but at an angle of 60° to get it to open. What is the torque (direction and magnitude) about the hinge if Nanook pushes 8 in from the hinge? Simplify your answer.

Viewpoint is from the swallow nesting above the door:



Solution: For P at the hinge and Q along the door 8 inches from P, since the torque is

$$\vec{\tau} = \overrightarrow{PQ} \times \vec{F}$$

then by the right hand rule, $\boxed{\text{the direction of the torque is out of the page towards you}}$. And the magnitude is:

$$\|\vec{\tau}\| = \left\|\overrightarrow{PQ}\right\| \left\|\vec{F}\right\| \sin 60° = \frac{8}{12}(30)\frac{\sqrt{3}}{2} = \boxed{10\sqrt{3} \text{ ft-lbs}}$$