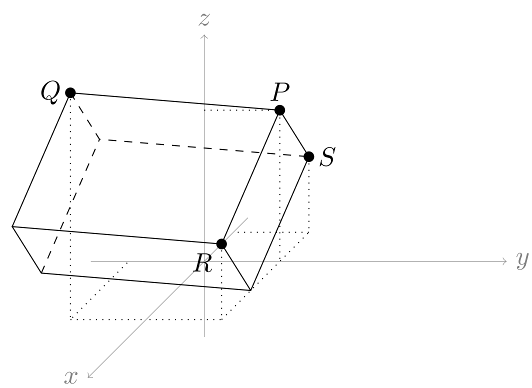
MATH202X-F01/UX1
Spring 2015

Practice Midterm Exam 1

Name: Answer Key

Instructions. You have 60 minutes. No calculators allowed. *Show all your work* in order to receive full credit.

1. Consider the points $P(0,1,2)$, $Q(2,-1,3)$, $R(2,1,1)$, and $S(-1,1,1)$ in space.



(a) Give the equation of the plane containing the points P, Q, and R.

Solution: A vector normal to the plane is orthogonal to any vector in the plane, such as for example $\overrightarrow{PQ}$, $\overrightarrow{PR}$. We have:

$$\overrightarrow{PQ} = \langle 2, -2, 1\rangle \quad , \quad \overrightarrow{PR} = \langle 2, 0, -1\rangle .$$

And so a vector normal to the plane is

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & -2 & 1 \\ 2 & 0 & -1 \end{vmatrix} = \langle 2, 4, 4\rangle .$$

We can take any scalar multiple of the above. A judicious choice is: $\langle 1, 2, 2\rangle$. And the equation of the plane is of the form:

$$x + 2y + 2z + d = 0.$$

We now substitute the coordinates of one of the given points (P, Q, or R) in the above to find:

$$0 + 2(1) + 2(2) + d = 0 \quad \Longrightarrow \quad d = -6,$$

and the equation of the plane is:

$$\boxed{x + 2y + 2z - 6 = 0.}$$

(b) Find the volume of the parallelepiped with vertices P, Q, R and S as drawn above.

Solution: The parallelepiped has adjacent edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ so its volume is given by:

$$\boxed{V = \left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = |\langle -1, 0, -1\rangle \cdot \langle 2, 4, 4\rangle| = |-1(2) + 0(4) - 1(4)| = |-6| = 6.}$$

2. Consider the following planes in space:

$$\begin{array}{ll} \text{Plane 1} & x + y - z - 2 = 0 \\ \text{Plane 2} & 2x - y + z - 1 = 0 \end{array}$$

(a) Find the point with $y-$coordinate equal to zero which lies on both planes.

Solution: We need to solve the system:

$$\begin{cases} x - z = 2 \\ 2x + z = 1 \end{cases}.$$

Adding up the two equations we find $3x = 3$, hence $x = 1$. And plugging that value back in the first (or second) equation one gets $z = -1$. Hence the point is: $\boxed{(1, 0, -1)}$.

(b) Give parametric equations for the line of intersection of the two planes.

Solution: The direction of the line of intersection is given by the cross product of the normal vectors to the planes:

$$\langle 1, 1, -1 \rangle \times \langle 2, -1, 1 \rangle = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 0\vec{\imath} - 3\vec{\jmath} - 3\vec{k}.$$

We can use the scalar multiple $\langle 0, 1, 1 \rangle$ of the above for direction and the point $(1, 0, -1)$ found in the previous question to give the parametric equations of the line of intersection as follows:

$$\boxed{\begin{cases} x = 1 \\ y = t \\ z = -1 + t \end{cases}.}$$

3. Let $\vec{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$ be describing the motion of a particle along a plane curve over time.

(a) Find all the open intervals on which the curve is smooth.

Solution: The domain of $\vec{r}(t)$ is the whole real line, and $\vec{r'}(t) = \langle 1 - \cos t, \sin t\rangle$ is defined, and continuous also on the whole real line. But

$$\vec{r'}(t) = \vec{0} \quad \text{whenever} \quad t = 2n\pi$$

for n any integer. So by definition,

$$\boxed{\text{The curve described by } \vec{r}(t) \text{ is smooth on } \bigcup_{n\in\mathbb{Z}} (2n\pi, (2n+2)\pi)\,.}$$

(b) What is the speed of the particle at $t = \dfrac{\pi}{2}$?

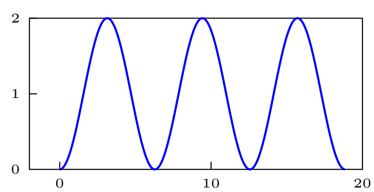
Solution: Since $\vec{r'}\left(\frac{\pi}{2}\right) = \left\langle 1 - \cos\frac{\pi}{2}, \sin\frac{\pi}{2}\right\rangle = \langle 1, 1\rangle$, then the speed of the particle at $t = \dfrac{\pi}{2}$ is:

$$\boxed{\left\|\vec{r'}\left(\frac{\pi}{2}\right)\right\| = \sqrt{2}.}$$

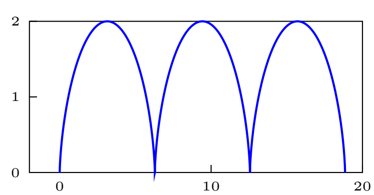
(c) Use smoothness to match the following plane curves to the correct graphs below.

$$\vec{r_1}(t) = \langle t - \sin t, 1 - \cos t\rangle \quad , \quad \vec{r_2}(t) = \langle t, 1 - \cos t\rangle \quad , \quad \vec{r_3}(t) = \langle 2t - \sin(2t), 1 - \cos t\rangle$$

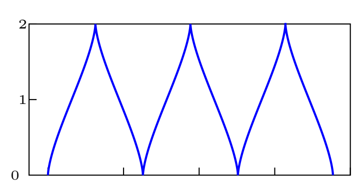
Briefly justify your answers. Each curve corresponds to $0 \le t \le 6\pi$ (and $x(t)$ is represented along the horizontal axis).



(a) $\vec{r_2}(t) = \langle t, 1 - \cos t\rangle$ **since** $\vec{r_2'} = \langle 1, \sin t\rangle \neq \vec{0}$ **on** $[0, 6\pi]$ **and this is the only smooth curve.**



(b) $\vec{r_1}(t) = \langle t - \sin t, 1 - \cos t\rangle$ **since we expect cusps at** $0, 2\pi, 4\pi, 6\pi$ **by our previous result.**



(c) $\vec{r_3}(t) = \langle 2t - \sin(2t), 1 - \cos t\rangle$ **since** $\vec{r_3'}(t) = \langle 2 - 2\cos(2t), \sin t\rangle = \vec{0}$ **for** $t = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$.

4. Time for some coordinate conversions!

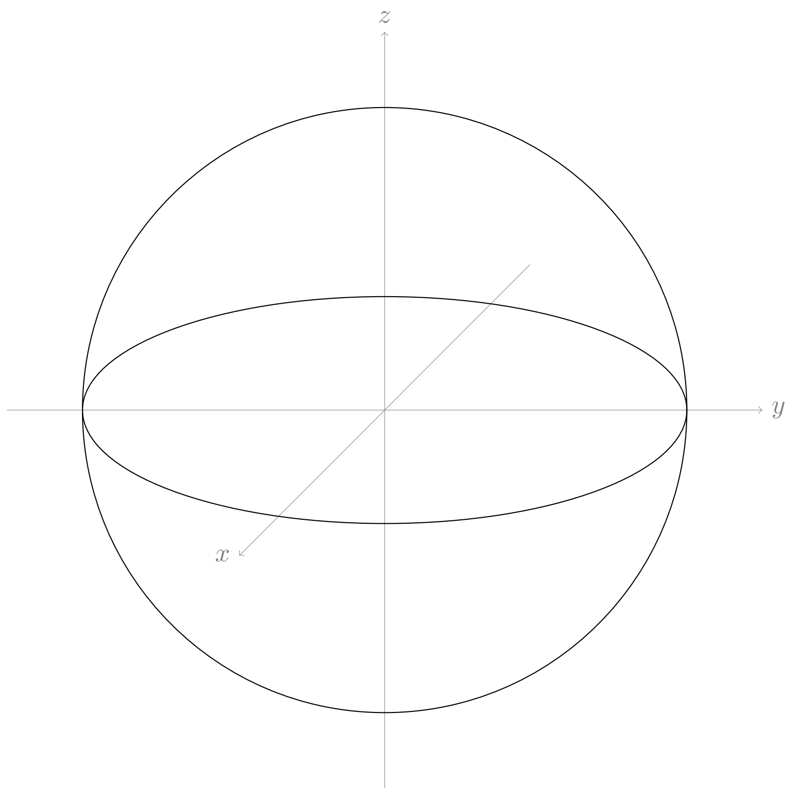
(a) Convert the point P with cylindrical coordinates $P\left(3, \frac{\pi}{6}, 1\right)$ to rectangular coordinates.

Solution: We have $r = 3$, $\theta = \frac{\pi}{6}$, and $z = 1$. So in rectangular coordinates,

$$\begin{cases} x = r\cos\theta = \frac{3\sqrt{3}}{2} \\ y = r\sin\theta = \frac{3}{2} \\ z = z = 1 \end{cases}, \text{ therefore } \boxed{P\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}, 1\right)} \text{ in rectangular coordinates.}$$

(b) Describe and sketch the surface in space whose equation in spherical coordinates is given by $\rho = 2$.

Solution:



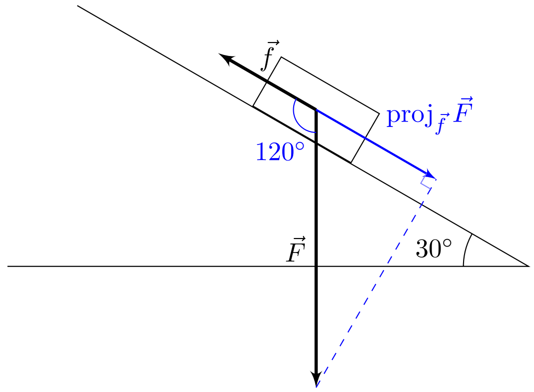
The equation $\rho = 2$ corresponds to $4 = \rho^2 = x^2 + y^2 + z^2$ in rectangular coordinates, i.e.

$$\boxed{\text{a sphere of radius 2 centered at the origin.}}$$

5. Little Jimmy is getting ready to go down the sledding hill, but he needs to adjust his hat amd scarf first. So he asks his little sister Tina to apply tension to the string by holding it in the same inclination as the hill.

(a) The force she exerts on it is represented by $\vec{f}$, and the force of gravity from Jimmy and his sled is represented by $\vec{F}$ in the sketch below. We assume no friction is present. Sketch $\text{proj}_{\vec{f}}\vec{F}$.

Solution:



(b) Assume the mass of little Jimmy and his sled together is $m = 50kg$ and that Tina exerts a force of magnitude $200N$. The hill is at an angle of 30° with the horizontal. Compute $\text{proj}_{\vec{f}}\vec{F}$. (Use $g = 9.8m/s^2$ for the magnitude of gravity.)

Solution: The force of gravity from Jimmy and his sled is:

$$\vec{F} = m\vec{g} \quad , \quad \left\|\vec{F}\right\| = 50 \times 9.8 = 490N.$$

The angle between $\vec{f}$ and $\vec{F}$ is 120°, so

$$\boxed{\text{proj}_{\vec{f}}\vec{F} = \frac{\vec{F}\cdot\vec{f}}{\left\|\vec{f}\right\|^2}\vec{f} = \frac{\left\|\vec{F}\right\|\left\|\vec{f}\right\|\cos\theta}{\left\|\vec{f}\right\|^2}\vec{f} = \frac{490 \times 200 \times \cos 120°}{200^2}\vec{f} = -1.225\vec{f}.}$$

(c) Based on your drawing and/or your result above, what will happen to the sled?

Solution: Since $\left\|\vec{f}\right\| < \left\|\text{proj}_{\vec{f}}\vec{F}\right\|$, Tina will not be able to hold on to the sled. It will start sliding down the hill.

6. Given the symmetry equations of two lines representing the trajectories of two particles (with t measured in seconds):

$$\text{Line 1:} \quad x+1 = \frac{y}{2} = \frac{z-1}{-1}$$
$$\text{Line 2:} \quad \frac{x-2}{-3} = y+1 = \frac{z+3}{4}$$

(a) Show that the graphs of the lines intersect at the point $(-1, 0, 1)$. Do the two particles collide at $(-1, 0, 1)$? Explain your answer.

Solution:

$$\text{Line 1:} \quad 0 = -1+1 = \frac{0}{2} = \frac{1-1}{-1}$$
$$\text{Line 2:} \quad 1 = \frac{-1-2}{-3} = 0+1 = \frac{1+3}{4}$$

The point $(-1, 0, 1)$ satisfies both sets of equations so it belongs to both lines.

However, the constant on the left represents the value of the parameter, i.e. the time t when the particle is at that point. So

they do not collide

since particle 1 is at $(-1, 0, 1)$ at $t = 0s$ while particle 2 gets there a second later.

(b) Find the distance from $(-1, 0, 1)$ to Plane 1 from problem 2: $x + y - z - 2 = 0$.

Solution: Since a normal vector to Plane 1 is $\langle 1, 1, -1\rangle$, then the distance is:

$$D = \frac{|(-1)+0-1-2|}{\|\langle 1,1,-1\rangle\|} = \frac{4}{\sqrt{3}}.$$

7. In the plane a particle moves with acceleration:

$$\vec{a}(t) = \pi^2 \cos(\pi t)\vec{\imath} + 3t^2\vec{\jmath}.$$

At time $t = 1$, its velocity is $\vec{v}(1) = \vec{\imath} - \vec{\jmath}$. At time $t = 0$, its position is $\vec{r}(0) = \vec{\imath} + \vec{\jmath}$. Find the position at time $t = 2$.

Solution: Let $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}$. Then

$$\begin{aligned}
& \vec{a}(t) = x''(t)\vec{\imath} + y''(t)\vec{\jmath} = \pi^2 \cos(\pi t)\vec{\imath} + 3t^2\vec{\jmath} \\
\Rightarrow\ & \vec{v}(t) = x'(t)\vec{\imath} + y'(t)\vec{\jmath} = \pi \sin(\pi t)\vec{\imath} + t^3\vec{\jmath} + \vec{C}_1 \\
& \vec{v}(1) = \vec{\jmath} + \vec{C}_1 = \vec{\imath} - \vec{\jmath} \\
\Rightarrow\ & \vec{C}_1 = \vec{\imath} - 2\vec{\jmath} \\
\Rightarrow\ & \vec{v}(t) = (\pi \sin(\pi t) + 1)\vec{\imath} + (t^3 - 2)\vec{\jmath} \\
\Rightarrow\ & \vec{r}(t) = (-\cos(\pi t) + t)\vec{\imath} + \left(\frac{t^4}{4} - 2t\right)\vec{\jmath} + \vec{C}_2 \\
& \vec{r_1}(0) = -\vec{\imath} + \vec{C}_2 = \vec{\imath} + \vec{\jmath} \\
\Rightarrow\ & \vec{C}_2 = 2\vec{\imath} + \vec{\jmath}
\end{aligned}$$

$$\Rightarrow \quad \vec{r}(t) = (t - \cos \pi t + 2)\vec{\imath} + \left(\frac{t^4}{4} - 2t + 1\right)\vec{\jmath}.$$

So the position at time $t = 2$ is:

$$\vec{r}(2) = (2 - \cos(2\pi) + 2)\vec{\imath} + (2^2 - 2(2) + 1)\vec{\jmath}.$$

$$\boxed{\vec{r}(2) = 3\vec{\imath} + \vec{\jmath}.}$$

8. An object moves along a trajectory so that its position $\vec{r}(t)$ as a function of time is given by:

$$\vec{r}(t) = \left\langle t^2, \frac{2t^3}{3}, t \right\rangle.$$

Find the tangential and normal components of the acceleration for the object. (*Hint:* Use a dot product formula for the tangential component, and deduce the normal one from $a_{\vec{N}} = \sqrt{\|\vec{a}\|^2 - a_{\vec{T}}^2}$.)

Solution:

$$\begin{aligned}
\vec{r}(t) &= \left\langle t^2, \frac{2t^3}{3}, t \right\rangle, \\
\vec{r'}(t) &= \left\langle 2t, 2t^2, 1 \right\rangle, \\
\vec{a}(t) = \vec{r''}(t) &= \langle 2, 4t, 0 \rangle, \\
\left\| \vec{r'}(t) \right\| &= \sqrt{4t^4 + 4t^2 + 1} = \sqrt{(2t^2+1)^2} = 2t^2 + 1, \\
\vec{T}(t) &= \frac{1}{2t^2+1} \left\langle 2t, 2t^2, 1 \right\rangle.
\end{aligned}$$

So we have that the tangential component of the acceleration is:

$$\boxed{a_{\vec{T}} = \vec{a} \cdot \vec{T} = \frac{4t}{2t^2+1} + \frac{8t^3}{2t^2+1} + 0 = \frac{4t(1+2t^2)}{2t^2+1} = 4t.}$$

And the normal component of the acceleration is:

$$\boxed{a_{\vec{N}} = \sqrt{\|\vec{a}\|^2 - a_{\vec{T}}^2} = \sqrt{\left(\sqrt{2^2 + (4t)^2}\right)^2 - (4t)^2} = \sqrt{4 + 16t^2 - 16t^2} = 2.}$$

9. In this problem, we deal with the same object as in the previous problem, that is whose trajectory can be described by the position vector:

$$\vec{r}(t) = \left\langle t^2, \frac{2t^3}{3}, t \right\rangle.$$

You should be able to use some of your work for these new questions.

(a) Find the arc length function of the trajectory (assume $t \geq 0$), and use it to find the distance traveled between $t = 0$ and $t = 2$ along the trajectory.

Solution:

$$\boxed{s(t) = \int_0^t \left\| \vec{r'}(u) \right\| du = \int_0^t (2u^2 + 1) du = \frac{2}{3}t^3 + t.}$$

And so the distance traveled is:

$$\boxed{s(2) = \frac{22}{3}.}$$

(b) Find the curvature K of the trajectory at time t using the formula of your choice.

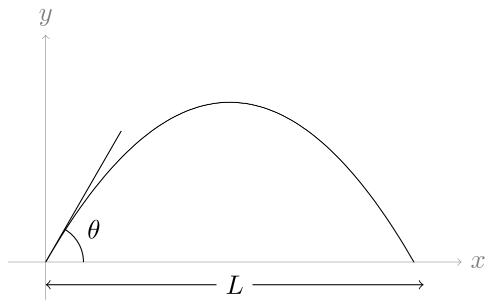
Solution: A really fast computation is by using the formula:

$$a_{\vec{N}} = K \left(\frac{ds}{dt} \right)^2.$$

Then

$$\boxed{K = \frac{a_{\vec{N}}}{\left\| \vec{r'}(t) \right\|^2} = \frac{2}{(2t^2 + 1)^2}.}$$

10. A projectile is fired from ground level at an angle θ with the horizontal. The projectile is to have a range of L units. Find the initial speed necessary. (Your final expression will contain θ and L.)



Solution: Let $\vec{r}(t)$ be the position vector, $\vec{v}(t)$ the velocity, and $\vec{a}(t)$ the acceleration. At $t = 0$, the projectile is fired from the ground, i.e. $\vec{r}(0) = \langle 0, 0 \rangle$ and at t_1, the projectile lands on the ground, i.e. $\vec{r}(t_1) = \langle L, 0 \rangle$ (for the initial speed/velocity we're looking for).
Note also that the acceleration is constant and due to gravity: $\vec{a}(t) = \langle 0, -g \rangle$ where $g = 9.8m/s^2$. We also have

$$\vec{v}(0) = \langle v_0 \cos\theta, v_0 \sin\theta \rangle$$

where v_0 is he initial speed ($v_0 = \|\vec{v}(0)\|$). So

$$\begin{aligned} &\vec{v}(t) = \langle v_0 \cos\theta, v_0 \sin\theta - gt \rangle \\ \Rightarrow \quad &\vec{r}(t) = \left\langle v_0 t \cos\theta, v_0 t \sin\theta - \frac{1}{2} g t^2 \right\rangle \end{aligned}$$

Since $\vec{r}(t_1) = \langle L, 0 \rangle$, we have:

$$\begin{cases} v_0 t_1 \cos\theta = L \\ v_0 t_1 \sin\theta - \frac{1}{2} g t_1^2 = 0 \end{cases}.$$

From the first coordinate we see that $t_1 \neq 0$ for any $L \neq 0$ (and for $L = 0$, we can take $\vec{v}(0) = 0$) and so the second coordinate gives us:

$$t_1 = \frac{2 v_0 \sin\theta}{g}.$$

Plugging this back into the first coordinate, we have:

$$L = \frac{2 v_0 \sin\theta}{g} v_0 \cos\theta = \frac{v_0^2 \sin(2\theta)}{g}$$

and since $0 < \theta < \pi/2$, the initial speed is

$$\boxed{v_0 = \sqrt{\frac{Lg}{\sin(2\theta)}}.}$$