Exam 1 Name: Solutions
Math 202 F01 10/03/12

1. (15 pts.–5 pts. each) A plane is given by the equation $3x - 2y + 5z = 1$.

   (a) Give an equation for the parallel plane through the point $(3, 1, 2)$.

   Alternate solution: $3(x-3)-2(y-1)+5(z-2)=0$

   Parallel planes have equations $3x-2y+5z=d$.
   To pick d so $(3,1,2)$ is on the plane we need $3\cdot 3-2\cdot 1+5\cdot 2=d$
   $17=d$
   so $3x-2y+5z=17$

   (b) Give a parameterization of the line through the point $(3, 1, 2)$ that is orthogonal to the plane.

   direction $\langle 3,-2,5\rangle$
   point $(3,1,2)$
   $\vec{r}(t)=\langle 3+3t,\ 1-2t,\ 2+5t\rangle$

   (c) What is the angle between this plane and the xz-coordinate plane? (Your answer may involve an inverse trigonometric function.)

   normal to plane above: $\vec{n}=\langle 3,-2,5\rangle$
   normal to xz-plane : $\vec{m}=\langle 0,1,0\rangle$

   $\|\vec{n}\|\,\|\vec{m}\|\cos\theta=\vec{n}\cdot\vec{m}$

   $\sqrt{9+4+25}\cdot 1\ \cos\theta=-2$

   $\cos\theta=\frac{-2}{\sqrt{38}}=-\sqrt{\frac{2}{19}}$

   $\theta=\cos^{-1}\left(-\sqrt{\frac{2}{19}}\right)$

   Note: $\theta=\cos^{-1}\left(\sqrt{\frac{2}{17}}\right)$ is also correct, since the problem did not specify the acute or obtuse angle

2. (14 pts.–7 pts. each) Due to gravity, an object that weighs 5 N slides down a straight frictionless ramp from the point $(0, 0, 10)$ m to the point $(0, 30, 0)$ m.

   (a) How much work was done by gravity? (Include appropriate units in your answer.)

   $\vec{F}=\langle 0,0,-5\rangle\ N$
   $\vec{d}=\langle 0,30,0\rangle-\langle 0,0,10\rangle=\langle 0,30,-10\rangle\ m$

   $W=\vec{F}\cdot\vec{d}=50\ Nm$

(b) Give a vector describing a force along the ramp that would have prevented the object from moving.

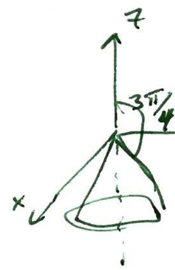
$$-\underbrace{\text{proj}_{\vec{d}}\vec{F}} = -\frac{\vec{F}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d} = -\frac{50}{0+900+100}\langle 0, 30, -10\rangle$$

$$\boxed{= \langle 0, -3/2, \tfrac{1}{2}\rangle \text{ N}}$$

This opposes the part of $\vec{F}$ that points along the ramp

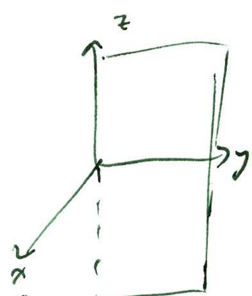
3. (20 pts.–5 pts. each)Roughly sketch the following surfaces in 3-d, given by equations in various coordinate systems. Include the x-,y-, and z-axes in each sketch.

(a) $\phi = \frac{3\pi}{4}$



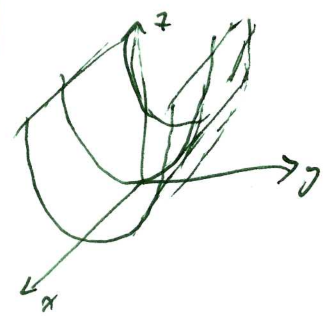
cone opening downward

(b) $\theta = \frac{\pi}{2}$



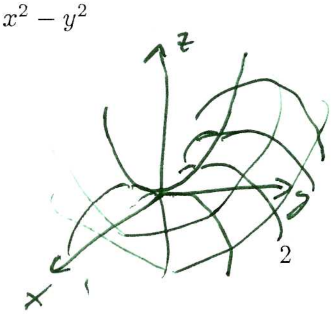
y-z half-plane with $y \geq 0$

(c) $y^2 - z = 0$



parabolic cylinder

(d) $z = x^2 - y^2$



saddle (up in $\pm x$ directions down in $\pm y$ directions)

4. (14 pts.) An object moves through space with acceleration vector $\mathbf{a}(t) = \langle t, \pi\sin(\pi t), -2\rangle\ m/sec^2$. At time $t = 0$, its velocity is $\mathbf{v}(0) = \langle 0, 0, 4\rangle$.

(a) (7 pts.) Find the object's velocity $\mathbf{v}(t)$, as a function of time.

$$\vec{v}(t) = \langle \tfrac{t^2}{2} + c_1,\ -\cos(\pi t) + c_2,\ -2t + c_3 \rangle$$

$$\vec{v}(0) = \langle c_1,\ -(1) + c_2,\ c_3 \rangle = \langle 0, 0, 4\rangle$$

so

$$\vec{v}(t) = \langle \tfrac{t^2}{2},\ 1 - \cos(\pi t),\ 4 - 2t \rangle$$

(b) (3 pts.) What is the object's speed at time $t = 1$?

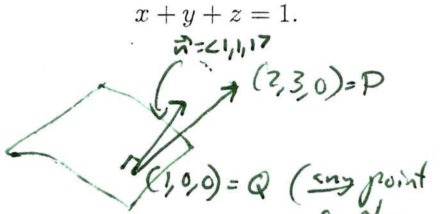
$$\vec{v}(1) = \langle \tfrac{1}{2},\ 1-(-1),\ 4-2 \rangle = \langle \tfrac{1}{2}, 2, 2\rangle$$

$$\|\vec{v}(1)\| = \sqrt{\tfrac{1}{4} + 4 + 4} = \sqrt{\tfrac{33}{4}} = \frac{\sqrt{33}}{2}\ \frac{m}{sec}$$

(c) (4 pts.) Give an integral for the total distance the object travels between time $t = 0$ and $t = 4$. (Do not evaluate the integral, but leave it in a form where only single-variable calculus is needed to understand it.)

$$\int_0^4 \|\vec{v}(t)\|\,dt = \int_0^4 \sqrt{\left(\tfrac{t^2}{2}\right)^2 + (1-\cos(\pi t))^2 + (4-2t)^2}\,dt$$

5. (11 pts.) Find the distance between the point $(2, 3, 0)$ and the plane $x + y + z = 1$.



$$\mathrm{proj}_{\vec{n}} \overrightarrow{QP} = \frac{\vec{n}\cdot\overrightarrow{QP}}{\vec{n}\cdot\vec{n}}\vec{n}$$

$$= \frac{\langle 1,1,1\rangle \cdot \langle 1,3,0\rangle}{\langle 1,1,1\rangle \cdot \langle 1,1,1\rangle}\langle 1,1,1\rangle$$

$$= \frac{4}{3}\langle 1,1,1\rangle$$

so $\|\mathrm{proj}_{\vec{n}}\overrightarrow{QP}\| = \frac{4}{3}\sqrt{3}$

6. (12 pts.–6 pts. each) Consider the 3 vectors

$$\mathbf{a} = \langle 1, 1, 0 \rangle,$$
$$\mathbf{b} = \langle 2, 1, 1 \rangle,$$
$$\mathbf{c} = \langle 1, 0, 5 \rangle.$$

(a) Compute $\mathbf{b} \times \mathbf{c}$, and state the geometric meaning of what you have computed.

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 0 & 5 \end{vmatrix} = (5-0)\hat{i} - (10-1)\hat{j} + (0-1)\hat{k} = \langle 5, -9, -1 \rangle$$

This is a vector $\perp$ to $\vec{b}, \vec{c}$ (given by the "right hand rule") whose length is the area of a parallelogram with sides $\vec{b}, \vec{c}$

(b) Compute $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ and state the geometric meaning of what you have computed.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \langle 1, 1, 0 \rangle \cdot \langle 5, -9, -1 \rangle = 5 - 9 + 0 = -4$$

This is $\pm$(the volume of a parallelopiped with edges $\vec{a}, \vec{b}, \vec{c}$)

7. (14 pts.) Consider the parameterized path $\mathbf{r}(t) = \ln t\, \mathbf{i} + (t+2)\, \mathbf{j}$.

(a) (6 pts.) Compute the unit tangent vector $\mathbf{T}(t)$.

$$\vec{r}'(t) = \langle \tfrac{1}{t}, 1 \rangle$$

$$\vec{T}(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{\frac{1}{t^2}+1}} \langle \tfrac{1}{t}, 1 \rangle = \frac{t}{\sqrt{1+t^2}} \langle \tfrac{1}{t}, 1 \rangle$$

$$\vec{T}(t) = \frac{1}{\sqrt{1+t^2}} \langle 1, t \rangle$$

(b) (6 pts.) Compute the unit normal vector $\mathbf{N}(t)$.

$$\vec{T}'(t) = -\tfrac{1}{2}(1+t^2)^{-3/2}(2t)\langle 1, t \rangle + (1+t^2)^{-1/2} \langle 0, 1 \rangle$$
$$= (1+t^2)^{-3/2}\left[-t\langle 1, t \rangle + (1+t^2)\langle 0, 1 \rangle\right]$$
$$= (1+t^2)^{-3/2} \langle -t, 1 \rangle$$

normalizing $\langle -t, 1 \rangle$ yields

$$\vec{N}(t) = \frac{1}{\sqrt{t^2+1}} \langle -t, 1 \rangle$$

(c) (2 pts.) What does $\mathbf{N}(t)$ tell us about an object following the path?

The direction $\perp$ to $T(t)$ in which the object is turning.