Math F253   Midterm 1   Fall 2024

Name: ______________________   **Section: 901 (Maxwell)**

Student Id: ______________________

Rules:

You have 70 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

A scientific or graphing calculator is allowed.

A one page sheet of paper (8 1/2 in. x 11 in.) with handwritten notes on one side is allowed.

No other aids are permitted.

Include **units** in your answer whenever appropriate.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

If you need extra space, you can use the back sides of the pages. Please make it obvious when you have done so.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	18	
2	10	
3	10	
4	10	
5	10	
6	8	
Extra Credit	5	
Total	64	

1. (18 points)

Consider the following two lines:

$$\mathbf{r}_1(t) = \langle 1, 2, 3\rangle + t\,\langle 1, 1, 2\rangle$$
$$\mathbf{r}_2(t) = \langle 1, 2, 3\rangle + t\,\langle -1, 2, -1\rangle$$

a. Find a point that lies on both lines.

$$\vec{r}_1(0) = \vec{r}_2(0) = \boxed{\langle 1, 2, 3\rangle}$$

b. Compute the angle between the two lines.

$$\vec{v}_1 = \langle 1, 1, 2\rangle \qquad \vec{v}_1 \cdot \vec{v}_2 = -1 + 2 - 2 = -1$$
$$\vec{v}_2 = \langle -1, 2, -1\rangle \qquad \|\vec{v}_1\| = \sqrt{1+1+4} = \sqrt{6}$$
$$\|\vec{v}_2\| = \sqrt{1+4+1} = \sqrt{6}$$

$$\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\|\,\|\vec{v}_2\|} = \frac{-1}{6} \Rightarrow \theta = \boxed{\operatorname{acos}\left(-\tfrac{1}{6}\right) \approx 99.6^\circ}$$

c. Compute the distance between $\mathbf{r}_1(1)$ and $\mathbf{r}_2(1)$. Note that $t = 1$ in these expressions.

$$\vec{r}_1(1) = \langle 2, 3, 5\rangle \qquad \vec{r}_1(1) - \vec{r}_2(1) = \langle 2, -1, 3\rangle$$

$$\vec{r}_2(1) = \langle 0, 4, 2\rangle \qquad \|\vec{r}_1(1) - \vec{r}_2(1)\| = \sqrt{4+1+9}$$

$$\boxed{= \sqrt{14}}$$

Problem 1 continued....

Continued....

Recall:

$$\mathbf{r}_1(t) = \langle 1,2,3\rangle + t\langle 1,1,2\rangle$$
$$\mathbf{r}_2(t) = \langle 1,2,3\rangle + t\langle -1,2,-1\rangle$$

d. Compute an equation for the plane that contains both lines.

$\vec{v}_1 = \langle 1,1,2\rangle$

$\vec{v}_2 = \langle -1,2,-1\rangle$

$\vec{n} = \vec{v}_1 \times \vec{v}_2$

$$\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ -1 & 2 & -1 \end{array} \rightarrow (-1-4)\hat{i} - (-1+2)\hat{j} + (2+1)\hat{k}$$

$\Rightarrow \vec{n} = \langle -5,-1,3\rangle$

$$\boxed{-5(x-1) - (y-2) + 3(z-3) = 0}$$

e. Determine, with justification, if the point $\langle 1,5,4\rangle$ lies on this plane.

$-5(1-1) - (5-2) + 3(4-1) = 0 - 3 + 3 = 0$ ✓

$\boxed{\text{on plane}}$

2. (10 points)

Consider the curve

$$\mathbf{r}(t) = \langle e^t, e^{-t} \rangle .$$

a. Determine $\mathbf{r}(0)$ and $\mathbf{r}'(0)$.

$\vec{r}\,'(t) = \langle e^t, -e^{-t} \rangle$

$\vec{r}(0) = \langle e^0, e^0 \rangle = \langle 1, 1 \rangle$

$\vec{r}\,'(0) = \langle 1, -1 \rangle$

b. Compute the equation of a line $\ell(t)$ with $\ell(0) = \mathbf{r}(0)$ and $\ell'(0) = \mathbf{r}'(0)$. That is, compute the tangent line to $\mathbf{r}(t)$ at time $t = 0$.

$\ell(t) = \langle 1, 1 \rangle + t\langle 1, -1 \rangle$

c. Use the tangent line to estimate the value of $\mathbf{r}(1/2)$. No credit will be granted for computing $\mathbf{r}(1/2)$ here; you must use the tangent line.

$\ell(1/2) = \langle 1, 1 \rangle + \frac{1}{2}\langle 1, -1 \rangle$

$= \langle 3/2, 1/2 \rangle$

d. Compute the size of the error of your approximation. Keep in mind that your approximation is a vector, but the size of error is a scalar. (A symbolic expression is acceptable, but you may wish to compute the value with your calculator. That is, you are now authorized to compute $r(1/2)$!)

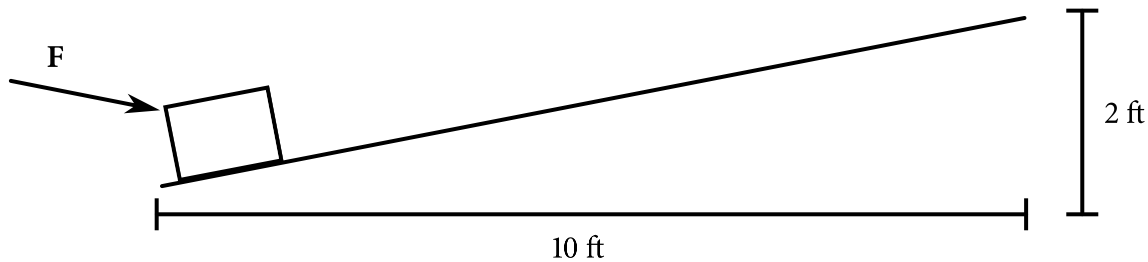
$\vec{r}(1/2) = \langle e^{1/2}, e^{-1/2} \rangle \approx \langle 1.64, 0.607 \rangle$

$\vec{r}(1/2) - \ell(1/2) = \langle e^{1/2} - \frac{3}{2}, e^{-1/2} - 1/2 \rangle \approx \langle 0.14, 0.107 \rangle$

$\| \vec{r}(1/2) - \ell(1/2) \| = \boxed{\left[\left(e^{1/2} - \frac{3}{2}\right)^2 + \left(e^{-1/2} - \frac{1}{2}\right)^2 \right]^{1/2} \approx 0.176}$

3. (10 points)

You are pushing a box up the entire length of a ramp that rises 2 feet over a distance of 10 feet.



a. You push with a force **F** with magnitude 20 pounds and at an angle of 15° below horizontal. Compute the total work done by you moving the box up the entire ramp.

$\vec{F} = 20\cos(15°)\hat{i} - 20\sin(15°)\hat{j}$ lb

$\vec{v} = \langle 10, 2 \rangle$ ft

work: $\vec{F}\cdot\vec{v} = \boxed{(200\cos(15°) - 40\sin(15°)) \text{ ft lb} \approx 182.8 \text{ ft lb}}$

b. The vector $\mathbf{v} = \langle 10, 2\rangle$ ft points in the direction of the slope of the ramp. Compute the projection $\text{proj}_{\mathbf{v}}\mathbf{F}$ of the force in the direction up the ramp.

$\text{proj}_{\vec{v}}\vec{F} = \frac{\vec{F}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$

$\|\vec{v}\|^2 = 104 \text{ ft}^2$

$\text{proj}_{\vec{v}}\vec{F} = \boxed{\frac{(200\cos(15°) - 40\sin(15°))}{104}\langle 10, 2\rangle \text{ lb}}$

$\approx \frac{182.8}{104}\langle 10, 2\rangle$ lb

$\approx \boxed{\langle 17.6, 3.52\rangle \text{ lb}}$

c. How much force pointing down the ramp (e.g., friction) would prevent you from moving the box? You answer should be a scalar.

$\frac{\vec{F}\cdot\vec{v}}{\|\vec{v}\|} \approx \frac{182.8}{\sqrt{104}} = 17.9$ lb

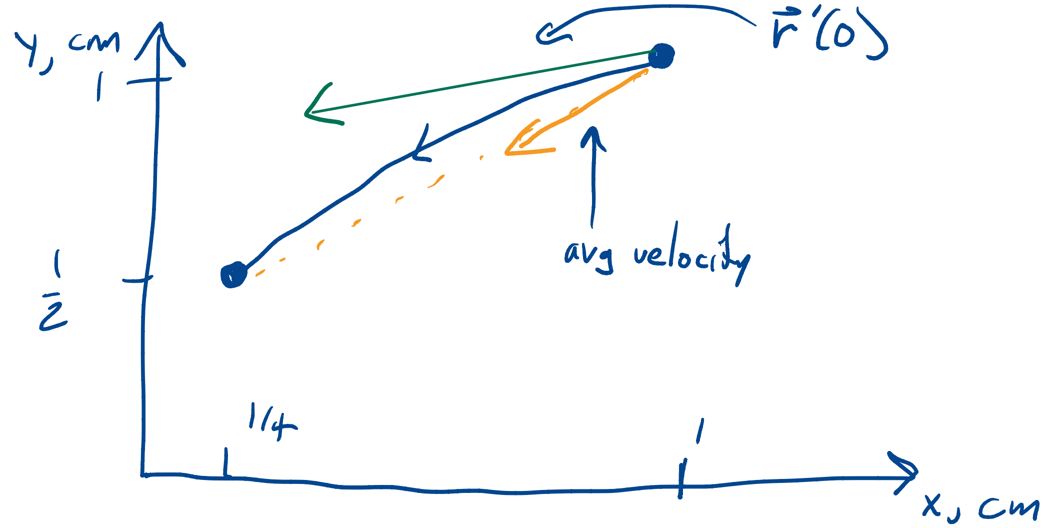
(or use norm of)

4. (10 points)

An object moves during the time interval $1 \le t \le 2$ seconds with the following position vector:

$$\mathbf{r}(t) = \left\langle 1/t^2, 1/t \right\rangle \text{ cm.}$$

a. Sketch the graph of the curve traversed by the object for whole interval $1 \le t \le 2$. Your diagram should include an indication of which direction the curve is traversed as t increases. It might be helpful to observe that $x = y^2$ on this curve. Make a nice, large figure, please!



b. Compute the average velocity from time $t = 1$ to $t = 2$. Add a vector that indicates this velocity to your diagram.

$$\vec{r}(2) - \vec{r}(1) = \left\langle \frac{1}{4}, \frac{1}{2} \right\rangle - \langle 1, 1 \rangle = \left\langle -\frac{3}{4}, -\frac{1}{2} \right\rangle \text{cm}$$

$$\vec{v}_{avg} = \frac{\vec{r}(2) - \vec{r}(1)}{2-1} = -\left\langle \frac{3}{4}, \frac{1}{2} \right\rangle \text{cm/s}$$

c. Compute the velocity of the object at time $t = 1$. Add this vector to your diagram.

$$\vec{r}\,'(t) = \langle -2t^{-3}, -t^{-2} \rangle \text{ cm/s}$$

$$\vec{r}\,'(1) = \langle -2, -1 \rangle \text{ cm/s}$$

d. Is the object traveling faster at $t = 1$ or $t = 2$? Provide a complete justification for full credit.

$$\vec{r}\,'(2) = \left\langle -\frac{2}{8}, -\frac{1}{4} \right\rangle = -\left\langle \frac{1}{4}, \frac{1}{4} \right\rangle \text{cm/s}$$

$$\boxed{\|\vec{r}\,'(1)\| = \sqrt{5} \text{ cm/s} \approx 1{,}5 \text{ cm/s}}$$ faster!

$$\vec{r}\,'(2) = \frac{\sqrt{2}}{4} \approx 0{,}35 \text{ cm/s}$$

5. (10 points)

Find the position $\mathbf{r}(t)$ of an object that has acceleration

$$\mathbf{r}''(t) = \langle -\cos(t), -\sin(t), e^{-t} \rangle \text{ m/s}^2$$

and that satisfies $\mathbf{r}(0) = \langle 0, 0, 1 \rangle$ m and $\mathbf{r}'(0) = \langle 0, 1, -2 \rangle$ m/s.

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), -e^{-t} \rangle + \vec{c}$$

$$\vec{r}\,'(0) = \langle 0, 1, -1 \rangle + \vec{c}$$

$$\vec{r}\,'(0) = \langle 0, 1, -2 \rangle \Rightarrow \vec{c} = \langle 0, 0, -1 \rangle$$

$$\boxed{\vec{r}\,'(t) = \langle -\sin(t), \cos(t), -1 - e^{-t} \rangle}$$

$$\vec{r}(t) = \langle \cos(t), \sin(t), -t + e^{-t} \rangle + \vec{d}$$

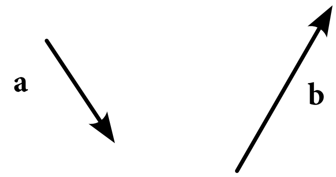
$$\vec{r}(0) = \langle 1, 0, 1 \rangle + \vec{d}$$

$$\vec{r}(0) = \langle 0, 0, 1 \rangle \qquad \Rightarrow \vec{d} = \langle -1, 0, 0 \rangle$$

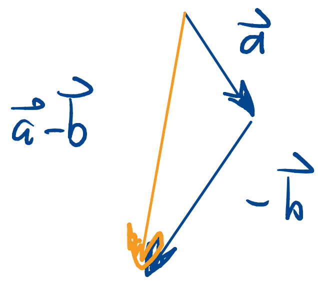
$$\boxed{\vec{r}(t) = \langle -1 + \cos(t), \sin(t), -t + e^{-t} \rangle}$$

6. (8 points)

Consider the vectors **a** and **b** depicted below.



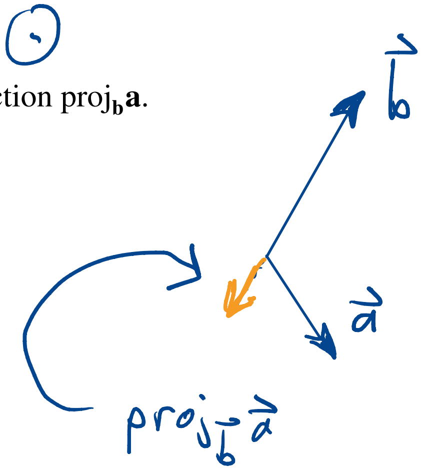
a. Sketch the vector **a** - **b**. Your sketch should contain enough details to know how your answer was arrived at.



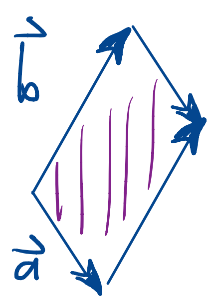
b. Determine if the cross product $\mathbf{a} \times \mathbf{b}$ is pointing into or out of the page.

out $\odot$

c. Sketch the projection $\text{proj}_{\mathbf{b}}\mathbf{a}$.



d. Sketch a region in the plane with an area equal to the magnitude of $\mathbf{a} \times \mathbf{b}$.



7. (Extra Credit: 5 points)

Earlier in the exam you worked with an object that has acceleration

$$\mathbf{r}''(t) = \langle -\cos(t), -\sin(t), e^{-t}\rangle$$

and that satisfies $\mathbf{r}(0) = \langle 0, 0, 1\rangle$ and $\mathbf{r}'(0) = \langle 0, 1, -2\rangle$.

Now suppose that a second object follows a path $\mathbf{s}(t)$ with $\mathbf{s}(0) = \langle -2, 1, 3\rangle$ and $\mathbf{s}'(0) = \langle -1, -4, 5\rangle$.

Compute $\dfrac{d}{dt}(\mathbf{s}(t) \times \mathbf{r}(t))$ at time $t = 0$.

$$\frac{d}{dt}\left(\vec{s}(t) \times \vec{r}(t)\right) = \vec{s}\,'(t) \times \vec{r}(t) + \vec{s}(t) \times \vec{r}\,'(t)$$

$\vec{s}\,'(0) \times \vec{r}(0)$:

$$\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -4 & 5 \\ 0 & 0 & 1 \end{matrix} \rightarrow -4\hat{i} + \hat{j}$$

$$= \langle -4, 1, 0\rangle$$

$\vec{s}(0) \times \vec{r}\,'(0)$

$$\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 3 \\ 0 & 1 & -2 \end{matrix} \rightarrow (-2-3)\hat{i} - (4)\hat{j} - 2\hat{k}$$

$$-5\hat{i} - 4\hat{j} - 2\hat{k}$$

$$= \langle -5, -4, -2\rangle$$

$$\langle -4, 1, 0\rangle + \langle -5, -4, -2\rangle$$

$$= \boxed{\langle -9, -3, -2\rangle}$$