Math F253 Midterm 1 Fall 2021

Name: Solutions

Section: F02 (Maxwell)

Student Id: ____________________

Calculator Model: ____________________

Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

A scientific or graphing calculator (without symbolic manipulation) is allowed.

A one page sheet of paper (8 1/2 in. x 11 in.) with handwritten notes on one side is allowed.

No other aids are permitted.

Place a box around your FINAL ANSWER to each question where appropriate.

If you need extra space, you can use the back sides of the pages. Please make it obvious when you have done so.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	12	
2	12	
3	12	
4	5	
5	12	
6	10	
7	12	
Extra Credit	3	
Total	75	

1. (12 points)

Consider the vectors $\mathbf{v} = \langle 1, -2, -4 \rangle$ and $\mathbf{w} = \langle 2, 3, -2 \rangle$.

a. Is the angle between the two vectors acute, obtuse, or right? Justify your answer.

$$1 \cdot 2 + (-2) \cdot 3 + (-4) \cdot (-2) = 2 - 6 + 8 = 4 > 0$$

$$\boxed{\text{acute}}$$

b. Find a unit vector pointing in the same direction as $\mathbf{w}$.

$$|\vec{w}|^2 = 2^2 + 3^2 + (-2)^2 = 4 + 9 + 4 = 17$$

$$\frac{\vec{w}}{|w|} = \left\langle \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{-2}{\sqrt{17}} \right\rangle$$

c. Compute the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -4 \\ 2 & 3 & -2 \end{vmatrix} = \hat{i}(4+12) - \hat{j}(-2+8) + \hat{k}(3+4)$$

$$= 16\hat{i} - 6\hat{j} + 7\hat{k}$$

$$|\vec{v} \times \vec{w}| = \sqrt{16^2 + (-6)^2 + 7^2}$$

$$= \sqrt{341}$$

$$\boxed{\text{area: } \sqrt{341}}$$

2. (12 points)

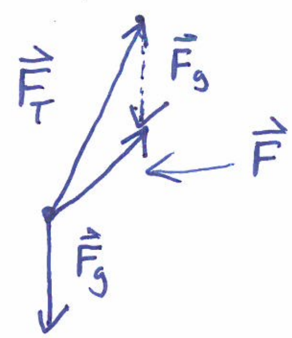
A rocket experiences two forces in units of Newtons:

- gravitational, $\mathbf{F}_g = -2000\mathbf{k}$
- thrust, $\mathbf{F}_T = 1000\mathbf{i} + 4000\mathbf{k}$

a. Compute the total force $\mathbf{F}$ the rocket experiences.

$$\vec{F} = \vec{F}_g + \vec{F}_T = \left(1000\,\hat{i} + 2000\,\hat{k}\right) N$$

b. Make a diagram with arrows that illustrates both the two forces $\mathbf{F}_g$ and $\mathbf{F}_T$ as well as the total force $\mathbf{F}$. Your diagram should be reasonably accurate.



c. In a short time interval the rocket experiences the constant total force $\mathbf{F}$ you computed in part **a** as it moves from $P(10, 5, 20)$ to $Q(13, 5, 26)$; all units are in meters. Compute the work done on the rocket by gravity and thrust combined.

$$\vec{PQ} = \langle 3, 0, 6 \rangle$$

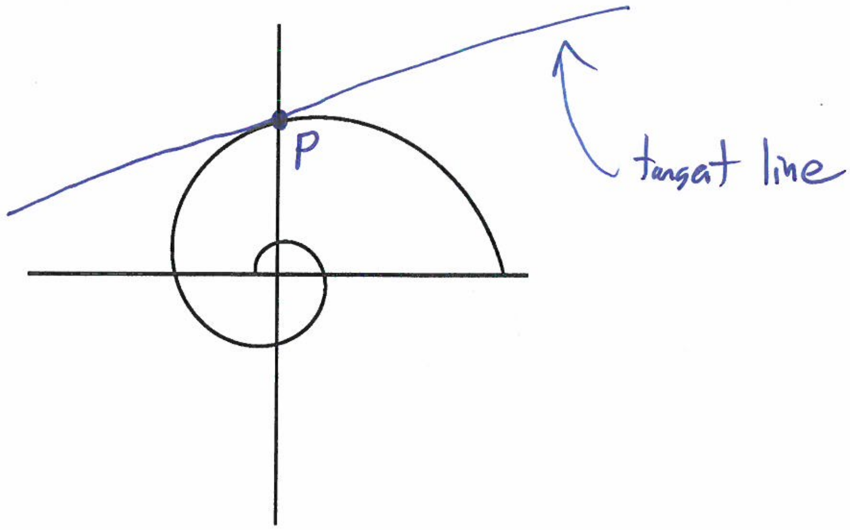
work: $\vec{F} \cdot \vec{PQ} = \langle 1000, 0, 2000 \rangle \cdot \langle 3, 0, 6 \rangle$

$= 3000 + 0 + 12000$

$= 15000\ \underbrace{N\,m}_{\to\, = J}$

3. (12 points)

The figure below contains a sketch of the curve parameterized by $\mathbf{r}(t) = \left\langle e^{-t/4}\cos(t), e^{-t/4}\sin(t)\right\rangle$ for $0 \le t \le 3\pi$.



a. Label the point $P = \mathbf{r}(\pi/2)$ in the diagram.

b. Sketch the tangent **line** at $t = \pi/2$.

c. Find a formula for the tangent line $\vec{\ell}(s)$ at $t = \pi/2$.

$$\vec{r}(\pi/2) = \langle e^{-\pi/8}\cos(\pi/2), e^{-\pi/8}\sin(\pi/2)\rangle$$

$$= \langle 0, e^{-\pi/8}\rangle$$

$$\vec{r}\,'(t) = \left[-\tfrac{1}{4}e^{-t/4}\cos(t) - e^{-t/4}\sin(t)\right]\hat{\imath}$$

$$+ \left[-\tfrac{1}{4}e^{-t/4}\sin(t) + e^{-t/4}\cos(t)\right]\hat{\jmath}$$

$$\vec{r}\,'(\pi/2) = -e^{-\pi/8}\hat{\imath} - \tfrac{1}{4}e^{-\pi/8}\hat{\jmath}$$

$$\vec{\ell}(s) = \vec{r}(\pi/2) + s\,\vec{r}\,'(\pi/2)$$

$$= \langle 0, e^{-\pi/8}\rangle + s\langle -e^{-\pi/8}, -\tfrac{1}{4}e^{-\pi/8}\rangle$$

Continued....

$$= \langle se^{-\pi/8}, e^{-\pi/8} + (-\tfrac{1}{4})e^{-\pi/8}s\rangle$$

Problem 3 continued....

d. Find a parameter value t such that the tangent line to the curve $\mathbf{r}(t)$ is horizontal.

$$\vec{r}\,'(t) = \{\text{stuff}\}\,\hat{\imath} + \underbrace{\left[-\frac{1}{4}e^{-t/4}\sin(t) + e^{-t/4}\cos(t)\right]}_{\text{want this } = 0}\hat{\jmath}$$

$$-\frac{1}{4}e^{-t/4}\sin(t) + e^{-t/4}\cos(t) = 0$$

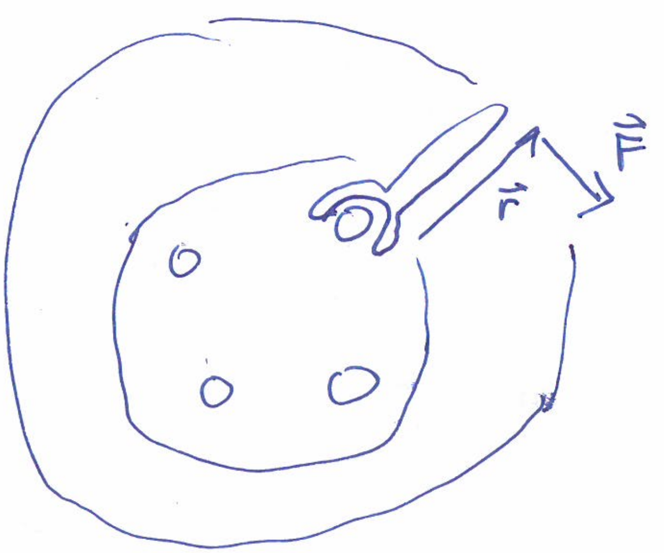
$$\cos(t) = \frac{1}{4}\sin(t)$$

$$\tan(t) = 4$$

$$t = \arctan(4) \approx 1.33$$

4. (5 points)

You are using a wrench to tighten a lug nut on a car tire. So you are turning the wrench clockwise. In what direction is the torque vector you are generating pointing? Explain.



$$\vec{r} \times \vec{F} = \vec{\tau}$$

into page by right-hand rule

5. (12 points)

Consider the plane $2x - 3y + 4z = 10$ and the line $\vec{\ell}(t) = \langle 1 + t, 1 - t, 3t \rangle$.

a. Find the point of intersection of the line and the plane.

$$2(1+t) - 3(1-t) + 4(3t) = 10$$

$$2 + 2t - 3 + 3t + 12t = 10$$

$$17t = 11$$

$$t = 11/17$$

$$\vec{\ell}(11/17) = \langle 1 + 11/17, 1 - 11/17, \tfrac{33}{17} \rangle = \langle \tfrac{28}{17}, \tfrac{6}{17}, \tfrac{33}{17} \rangle$$

b. Find the equation parallel to the plane given, but that passes through the point $P(2, 1, -1)$.

$$2(x-2) - 3(y-6) + 4(z+1) = 0$$

c. Show that the vector $\mathbf{v} = \langle 1, 6, 4 \rangle$ is parallel to the original plane.

$$\langle 2, -3, 4 \rangle \cdot \langle 1, 6, 4 \rangle = 2 - 18 + 16 = 0 \checkmark$$

d. Find a vector $\mathbf{w}$ that is not parallel to $\mathbf{v}$ but that is parallel to the original plane.

Use, e.g., $\vec{n} \times \vec{v}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 1 & 6 & 4 \end{vmatrix} = (-12 - 24)\hat{i} - (8 - 4)\hat{j} + (12 + 3)\hat{k}$$

$$= -36\hat{i} - 4\hat{j} + 15\hat{k}$$

6. (12 points)

Consider the curve

$$\mathbf{r}(t) = \left\langle \sqrt{2}\sin(4t), \cos(4t), \cos(4t) \right\rangle.$$

a. Compute $\mathbf{r}'(t)$

$$\vec{r}\,'(t) = \langle 4\sqrt{2}\cos(4t), -4\sin(4t), -4\sin(4t) \rangle$$

b. Find the speed of the curve as a function of t.

$$|\vec{r}\,'(t)|^2 = 16\cdot 2\cos^2(4t) + 16\sin^2(4t) + 16\sin^2(4t)$$
$$= 32\left(\cos^2(4t) + \sin^2(4t)\right)$$
$$= 32$$

$|\vec{r}\,'(t)| = \sqrt{32} = 4\sqrt{2}$ for all t

c. Compute the length of the curve from $t = 0$ to $t = \pi/4$.

$$\int_0^{\pi/4} |\vec{r}\,'(t)|\,dt = \int_0^{\pi/4} 4\sqrt{2}\,dt = 4\sqrt{2}\,t\Big|_0^{\pi/4}$$
$$= \sqrt{2}\,\pi$$

7. (10 points)

A particle moves with acceleration $\mathbf{a}(t) = \left\langle 1, \frac{t}{1+t^2}, 2t \right\rangle$ in units of $\mathrm{m/s^2}$. At $t = 0$ the particle has velocity $\mathbf{v}(0) = \langle 1, 0, -2 \rangle\,\mathrm{m/s}$.

a. Determine $\mathbf{v}(t)$.

$$\int 1\,dt = t$$

$$\int \frac{t}{1+t^2}\,dt = \int \frac{1}{2}\frac{1}{u}\,du = \frac{1}{2}\ln(|u|) = \frac{1}{2}\ln(1+t^2)$$

$$u = 1+t^2;\ du = 2t\,dt$$

$$\int 2t\,dt = t^2$$

$$\vec{v}(t) = \langle t, \tfrac{1}{2}\ln(1+t^2), t^2 \rangle + \vec{c}$$

$$\vec{v}(0) = \langle 1, 0, -2 \rangle$$

$$\vec{v}(0) = \langle 0, \tfrac{1}{2}\ln(1), 0^2 \rangle + \vec{c} = \vec{0} + \vec{c} \Rightarrow \vec{c} = \vec{v}(0)$$

$$\vec{v}(t) = \langle 1+t, \tfrac{1}{2}\ln(1+t^2), t^2-2 \rangle$$

b. Determine all times the velocity is horizontal.

z component zero: $t^2 - 2 = 0$

$t = \pm\sqrt{2}$

8. (Extra Credit: 3 points)

In problem **3a** the curve has a familiar shape and is contained in a plane. Find the shape and the plane.