MATH202X-F01/UX1 Practice Final Exam Name: Answer Key
Spring 2015

**Instructions.** (0 points) You have 120 minutes. Each problem is worth 10 points. No calculators allowed. *Show all your work* in order to receive full credit.

**1.** Consider the following three points: $A(-1,0,1)$, $B(1,1,2)$, and $C(1,2,0)$.

(a) Determine whether the three points are collinear.

*Solution*: $\overrightarrow{AB} = \langle 2,1,1\rangle$ ; $\overrightarrow{AC} = \langle 2,2,-1\rangle$. The vectors are not scalar multiples of each other i.e. $\overrightarrow{AB} \neq k\overrightarrow{AC}$ for any real number $k$, so $\boxed{A, B, C \text{ are not collinear.}}$

(b) If they are collinear, give the parametric equations of the line they form. If not, give the equation of the plane containing these three points.

*Solution*:

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 1 & 1 \\ 2 & 2 & -1 \end{vmatrix} \\ &= (-1-2)\vec{\imath} - (-2-2)\vec{\jmath} + (4-2)\vec{k} \\ &= \langle -3,4,2\rangle\end{aligned}$$

The vector $\langle -3,4,2\rangle$ is normal to the plane so the equation of the plane is given by (using the point $A$):

$$\begin{aligned}&-3(x+1)+4(y-0)+2(z-1)=0 \\ &-3x-3+4y+2z-2=0 \\ &\boxed{3x-4y-2z+5=0.}\end{aligned}$$

**2.** Consider the plane:

$$x+2y+3z+4=0$$

and the following symmetric equations for two distinct lines:

Line 1: $\dfrac{x+1}{2} = y = \dfrac{z+1}{-1}$,

Line 2: $x-1 = y-2 = \dfrac{z}{-1}$.

Classify the intersection of the plane with each of the lines. Is there a one-point intersection (if so, give the coordinates of the point), no intersection because the line is parallel to the plane, or is the line in the plane?

*Solution*: The normal vector to the plane is $\vec{n} = \langle 1,2,3\rangle$.

- Line 1: the direction vector is $\vec{v_1} = \langle 2,1,-1\rangle$ and since

$$\langle 1,2,3\rangle \cdot \langle 2,1,-1\rangle = 2+2-3 = 1 \neq 0,$$

then $\boxed{\text{Line 1 intersects the plane at one point.}}$ From the line we have $x = 2y-1$ and $z = -1-y$ so substituting into the plane equation:

$$(2y-1)+2y+3(-y-1)=0 \quad \Rightarrow \quad y=0$$

and so we have $x=-1$ and $z=-1$; hence $\boxed{(-1,0,-1)}$ is the point of interesection between Line 1 and the plane.

- Line 2: the direction vector is $\vec{v_2} = \langle 1, 1, -1\rangle$ and since

$$\langle 1, 2, 3\rangle \cdot \langle 1, 1, -1\rangle = 1 + 2 - 3 = 0,$$

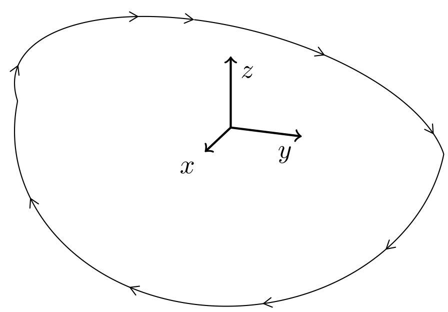
then Line 2 is either parallel to the plane (no intersection) or in the plane (infinitely many solutions). So we test a point from the line. For example from reading the equations, we can see that $(1, 2, 0)$ is a point on the plane and

$$1 + 2(2) + 3(0) + 4 = 9 \neq 0$$

so $\boxed{\text{Line 2 is parallel to the plane but not in it.}}$

**3.** Consider the following vector-valued function, representing the trajectory of a particle:

$$\vec{r}(t) = \sqrt{1 + \cos 2t}\,\vec{\imath} + 3\sin t\,\vec{\jmath} + 2\cos t\,\vec{k}.$$




(a) Find *all* the open interval(s) on which $\vec{r}(t)$ is smooth.

*Solution*: We have:

$$\vec{r}'(t) = \left\langle \frac{-\sin 2t}{\sqrt{1 + \cos 2t}}, 3\cos t, -2\sin t \right\rangle.$$

Note that $\vec{r}'(t)$ is continuous and nonzero wherever it is defined but since it's undefined for $t = (2k + 1\frac{\pi}{2})$ for any integer $k$, then

$$\boxed{\vec{r}(t) \text{ is smooth on } \bigcup_{k\in\mathbb{Z}} \left( \frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2} \right).}$$

(b) Find the speed of the particle at $t = 0$.

*Solution*: We have $\vec{r}'(0) = \langle 0, 3, 0\rangle$ so its speed is $\boxed{\left\|\vec{r}'(0)\right\| = 3.}$

Now, for extra credit: the parametric curve lies entirely on which of the following surface(s)? Check all that apply. You need not justify your answers.

- ☐ the ellipsoid: $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$,
- ☑ the hyperboloid of one sheet: $x^2 + \frac{y^2}{9} - \frac{z^2}{4} = 1$,
- ☑ the elliptic cylinder: $\frac{y^2}{9} + \frac{z^2}{4} = 1$,
- ☐ the hyperbolic paraboloid: $x = \frac{y^2}{9} - \frac{z^2}{4}$.

**4.** A particle in space moves with acceleration:

$$\vec{a}(t) = \left\langle 1, \frac{1}{2\sqrt{t}}, 0 \right\rangle \quad , \quad t \geq 1$$

such that its velocity at $t = 1$ is $\vec{v}(1) = \left\langle \frac{3}{2}, 1, \frac{\sqrt{3}}{2} \right\rangle$ and its position is $\vec{r}(1) = \left\langle 1, \frac{2}{3}, \sqrt{3} \right\rangle$.

(a) Find the position of the particle at $t = 4$.

*Solution*:

$$\vec{t} = \left\langle t, \sqrt{t}, 0 \right\rangle + \vec{C_1}$$

$$\left\langle \frac{3}{2}, 1, \frac{\sqrt{3}}{2} \right\rangle = \vec{v}(1) = \langle 1, 1, 0 \rangle + \vec{C_1} \quad \Rightarrow \quad \vec{C_1} = \left\langle \frac{1}{2}, 0, \frac{\sqrt{3}}{2} \right\rangle$$

$$\Rightarrow \quad \vec{t} = \left\langle t + \frac{1}{2}, \sqrt{t}, \frac{\sqrt{3}}{2} \right\rangle$$

$$\vec{r}(t) = \left\langle \frac{t^2}{2} + \frac{t}{2}, \frac{2}{3} t^{\frac{3}{2}}, \frac{t\sqrt{3}}{2} \right\rangle + \vec{C_2}$$

$$\left\langle 1, \frac{2}{3}, \sqrt{3} \right\rangle = \vec{r}(1) = \left\langle 1, \frac{2}{3}, \frac{\sqrt{3}}{2} \right\rangle + \vec{C_2} \quad \Rightarrow \quad \vec{C_2} = \left\langle 0, 0, \frac{\sqrt{3}}{2} \right\rangle$$

$$\Rightarrow \quad \vec{r}(t) = \left\langle \frac{t^2 + t}{2}, \frac{2}{3} t^{\frac{3}{2}}, \frac{\sqrt{3}}{2}(t+1) \right\rangle \quad \Rightarrow \quad \boxed{\vec{r}(4) = \left\langle 10, \frac{16}{3}, \frac{5\sqrt{3}}{2} \right\rangle.}$$

(b) Find the length of the curve between $t = 1$ and $t = 4$.

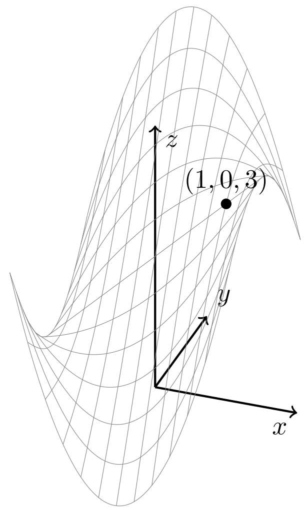
*Solution*:

$$\begin{aligned}
s = \int_1^4 \left\| \vec{r'}(t) \right\| \, dt &= \int_1^4 \sqrt{\left(t + \frac{1}{2}\right)^2 + t + \frac{3}{4}} \, dt \\
&= \int_1^4 \sqrt{t^2 + t + \frac{1}{4} + t + \frac{3}{4}} \, dt \\
&= \int_1^4 \sqrt{t^2 + 2t + 1} \, dt \\
&= \int_1^4 \sqrt{(1+t)^2} \, dt \\
&= \int_1^4 t + 1 \, dt \\
&= \frac{t^2}{2} + t \Big|_1^4 \\
&= 8 + 4 - \frac{1}{2} - 1 = 11 - \frac{1}{2} = \boxed{\frac{21}{2}}
\end{aligned}$$

**5.** Consider a point $P(1,0)$ in the domain of the surface

$$z = x\cos y - yx^2 + 2(y+1).$$

Assume the surface represents a hilly area, modeled below:




(a) What is the rate of change of altitude at the point $P$ when moving in the direction of the vector $\vec{v} = \langle 3, 4\rangle$?

*Solution*: Let $\vec{u}$ be the unit vector in the direction of $\vec{v}$:

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 3,4\rangle}{\sqrt{9+16}} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

Then the rate of change is:

$$\begin{aligned} \mathrm{D}_{\vec{u}} z|_{(1,0)} &= \nabla z \cdot \vec{u}|_{(1,0)} \\ &= \langle \cos y - 2xy, -x\sin y - x^2 + 2\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \Big|_{(1,0)} \\ &= \langle 1, 1\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\ &= \boxed{\frac{7}{5}.} \end{aligned}$$

(b) What is the direction of maximum decrease at $P$? I.e. if chased by a bear, which direction should $P$ take to get down that hill the fastest? What is the rate of decrease in that direction?

*Solution*:

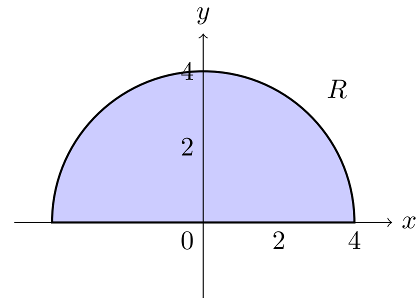
- direction: $-\nabla z(1,0) = \boxed{\langle -1, -1\rangle}$,
- rate of change: $-\|\nabla z(1,0)\| = \boxed{-\sqrt{2}}$.

**6.** Classify any critical points and then use Lagrange multipliers on the boundary and find the absolute maximum and minimum values of the function

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5$$

on the domain $x^2 + y^2 \leq 16$, $y \geq 0$.

*Solution*: The region is the upper part of the disk of radius 4:




We'll use the constraint $g(x, y) = x^2 + y^2$ and then $h(x, y) = y$. First to find critical points, we solve:

- $\nabla f = \vec{0}$: since $\nabla f = \langle 4x - 4, 6y \rangle$, we have:

$$\begin{cases} 4x - 4 = 0 \\ 6y = 0 \end{cases} \quad \Rightarrow \quad x = 1\ ,\ y = 0.$$

  The point $(1, 0)$ is in our region (barely because it's on the boundary) and since $f_{xx} = 4 > 0$, $f_{yy} = 6$, $f_{xy} = 0$, and $d = 4(6) - 0^2 > 0$ then $\boxed{f \text{ has a relative minimum at } (1, 0, -7).}$

- on the boundary $g(x, y) = 16$ (with $y \geq 0$): since $\nabla g = \langle 2x, 2y \rangle$, we have:

$$\nabla f = \lambda \nabla g \quad \Rightarrow \quad \begin{cases} 4x - 4 = 2\lambda x \\ 6y = 2\lambda y \end{cases} \quad \Rightarrow \quad \text{either } y = 0 \text{ or } \lambda = 3$$

  – if $y = 0$ then $x^2 + 0 = 16$ so $x = \ pm4$;
  – if $\lambda = 3$ then $4x - 4 = 6x$ so $x = -2$ and $(-2)^2 + y^2 = 16$ so $y^2 = 12$ and $y = 2\sqrt{3}$ (only solution satisfying $y \geq 0$).

- on the boundary $h(x, y) = 0$ (with $-4 \leq x \leq 4$): since $\nabla h = \langle 0, 1 \rangle$, we have:

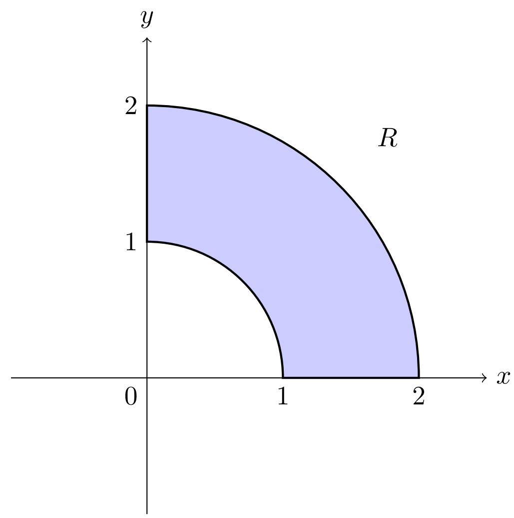
$$\nabla f = \lambda \nabla h \quad \Rightarrow \quad \begin{cases} 4x - 4 = 0 \\ 6y = \lambda \end{cases} \quad \Rightarrow \quad x = 1 \text{ and } y = 0$$

  which was found already through the search for critical points.

So putting all points of interest in a table:

| $x$ | $y$ | $z$ | |
|---|---|---|---|
| 1 | 0 | $-7$ | relative minimum and $\boxed{\text{absolute minimum}}$ |
| $-4$ | 0 | 43 | |
| 4 | 0 | 11 | |
| $-2$ | $2\sqrt{3}$ | 47 | $\boxed{\text{absolute maximum}}$ |

**7.** Find the moment of inertia about the $y-$axis $I_y$ for a planar lamina $R$ corresponding to the region below.

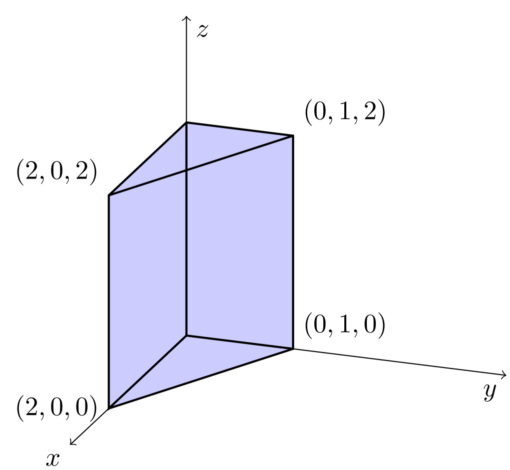



where the density $\rho(x, y) = y$.

*Solution*:

$$\begin{aligned} I_y &= \iint_R x^2 y \, dA = \int_0^{\frac{\pi}{2}} \int_1^2 r^2 \cos^2\theta \, r \sin\theta \, r \, dr \, d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_1^2 r^4 \cos^2\theta \sin\theta \, dr \, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left. \frac{r^5}{5} \right|_1^2 \cos^2\theta \sin\theta \, d\theta \\ &= \frac{31}{5} \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta \, d\theta = \frac{31}{5} \left[ -\frac{1}{3} \cos^3\theta \right]_0^{\frac{\pi}{2}} = \boxed{\frac{31}{15}} \end{aligned}$$

**8.** Consider the vector field $\overrightarrow{F}(x, y, z) = \left\langle x^2 y, xy^2, 2xyz \right\rangle$ acting on a closed surface $S$ consisting of the boundary of a triangular prism with the following vertices:

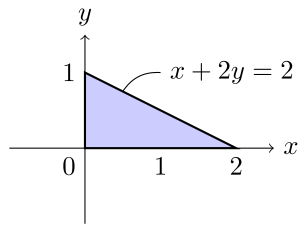


We are interested in evaluating the flux of the vector field over the surface $S$. Since the component functions of $\overrightarrow{F}$ have continuous first partial derivatives over the solid prism $Q$ , apply the Divergence Theorem

$$\underbrace{\oiint_S \overrightarrow{F}\cdot\overrightarrow{N}\,dS}_{\text{flux}} = \iiint_Q \operatorname{div}\overrightarrow{F}\,dV$$

to evaluate the flux indirectly.

*Solution*: If we use the order $dz\,dx\,dy$ for $Q$ we need its projection onto the $xy$-plane:




Now since the divergence of $\overrightarrow{F}$ is:

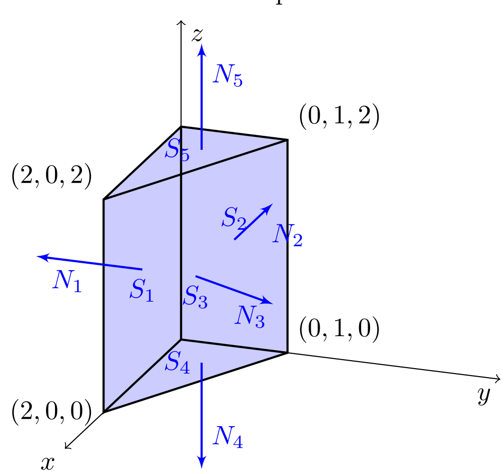
$$\operatorname{div}\overrightarrow{F} = 2xy + 2xy + 2xy = 6xy$$

then the flux is:

$$\begin{aligned}\iiint_Q \operatorname{div}\overrightarrow{F}\,dV &= \int_0^1\int_0^{2-2y}\int_0^2 6xy\,dz\,dx\,dy = \int_0^1\int_0^{2-2y} 12xy\,dx\,dy = \int_0^1 6x^2y\Big|_0^{2-2y}\,dy \\ &= \int_0^1 24y(1-y)^2\,dy = \begin{vmatrix} u = 8y & du = 8\,dy \\ dv = 3(1-y)^2\,dy & v = -(1-y)^3\end{vmatrix} \\ &= \cancelto{0}{\left[-8y(1-y)^3\right]_0^1} + \int_0^1 8(1-y)^3\,dy = -2(1-y)^4\Big|_0^1 = \boxed{2}.\end{aligned}$$

For extra credit, evaluate directly the flux. Note that you need to consider 5 surfaces separately including one which can be given by the following parametric representation: $\vec{r}(u,v) = \langle 2-2u, u, v\rangle$ for $0 \le u \le 1,\ 0 \le v \le 2$.

*Solution*: First let us label the five faces and their respective normal vectors:




Then note the few shortcuts along the way...

- on $S_1$, $y = 0$ so $\overrightarrow{F} = < x^2(0), x(0)^2, 2x(0)z > = \overrightarrow{0}$ along $S_1$ and therefore, $\iint_{S_1} \overrightarrow{F}\cdot\overrightarrow{N}\,dS = 0$;

- on $S_2$, $x = 0$ so $\overrightarrow{F} =< (0)^2 y, (0)y^2, 2(0)yz >= \overrightarrow{0}$ along $S_2$ and therefore, $\iint_{S_2} \overrightarrow{F} \cdot \overrightarrow{N}\, dS = 0$;
- on $S_3$, we can use the given parametrization $\vec{r}(u,v) = \langle 2-2u, u, v\rangle$ for $0 \le u \le 1,\ 0 \le v \le 2$. So

$$\overrightarrow{r_u} \times \overrightarrow{r_v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 1, 2, 0\rangle$$

  and since it is pointing outwards already, then we choose $\overrightarrow{N}\, dS = \langle 1, 2, 0\rangle\ dv\ du$ and so:

$$\begin{aligned}\iint_{S_3} \overrightarrow{F} \cdot \overrightarrow{N}\, dS &= \int_0^1 \int_0^2 \left\langle (2-2u)^2 u, (2-2u)u^2, 2(2-2u)uv\right\rangle \cdot \langle 1,2,0\rangle\ dv\ du \\ &= \int_0^1 \int_0^2 u(2-2u)^2 + 2u^2(2-2u) + 0\ dv\ du = \int_0^1 u(2-2u)\,(2-2u+2u)\,[v]_0^2\ du \\ &= \int_0^1 8u(1-u)\ du = \left[4u^2 - \frac{8u^3}{3}\right]_0^1 = 4 - \frac{8}{3} = \frac{4}{3};\end{aligned}$$

- on $S_4$, we have that $\overrightarrow{N_4} = \langle 0, 0, -1\rangle$ so only the $P$ component of $\overrightarrow{F}$ will survive; but $z = 0$ on $S_4$ so $P = 2xy(0) = 0$ and therefore, $\iint_{S_4} \overrightarrow{F} \cdot \overrightarrow{N}\, dS = 0$;
- on $S_5$, we have that $\overrightarrow{N_5} = \langle 0, 0, 1\rangle$ so only the $P$ component wil survive again; this time $z = 2$ and $P = 2xy(2) \neq 0$ everywhere so we need a representation of $S_4$; the easiest is to use the description of the face from the indirect computation done earlier (except now $z = 2$), i.e.

$$S_5 = \{(x, y, 2)\ :\ 0 \le y \le 1\ ,\ 0 \le x \le 2-2y\}$$

  and therefore,

$$\begin{aligned}\iint_{S_5} \overrightarrow{F} \cdot \overrightarrow{N}\, dS &= \int_0^1 \int_0^{2-2y} \left\langle x^2 y, xy^2, 4xy\right\rangle \cdot \langle 0,0,1\rangle\ dx\ dy = \int_0^1 \int_0^{2-2y} 4xy\ dx\ dy \\ &= \int_0^1 \left[2x^2 y\right]_0^{2-2y}\ dy = \int_0^1 2(2-y)^2 y\ dy \\ &= \int_0^1 8y(1-y)^2\ dy = \begin{vmatrix} u = 8y & du = 8\ dy \\ dv = (1-y)^2\ dy & v = -\frac{(1-y)^3}{3}\end{vmatrix} \\ &= \left[-8y\frac{(1-y)^3}{3}\right]_0^1 \xrightarrow{} 0 + \int_0^1 \frac{8}{3}(1-y)^3\ dy = -\frac{2}{3}(1-y)^4\Big|_0^1 = \frac{2}{3}.\end{aligned}$$

So putting them all together, we have:

$$\oiint_S \overrightarrow{F} \cdot \overrightarrow{N}\, dS = 0 + 0 + \frac{4}{3} + 0 + \frac{2}{3} = \boxed{2}.\quad \checkmark$$

Note that even though we get the same result, using the divergence theorem was much easier...

**9.** Consider a particle moving through space along the curve $C$ given by the following parametric representation:

$$\vec{r}(t) = \left\langle t^3 - 3t + 1, \frac{t}{2} + 1, \frac{t}{2}\cos \pi t\right\rangle \quad , \quad 0 \le t \le 2$$

and subject to the vector field: $\overrightarrow{F}(x,y,z) = \left\langle y^3 - 2xz, 3xy^2 + 2z, 2y - x^2\right\rangle$.

(a) Show that $\vec{F}$ is conservative.

*Solution*:

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y^3 - 2xz & 3x^2y + 2z & 2y - x^2 \end{vmatrix} = (2-2)\vec{\imath} - (-2x+2x)\vec{\jmath} + (3y^2 - 3y^2)\vec{k} = \vec{0}$$

so $\boxed{\vec{F} \text{ is conservative.}}$

(b) Find *all* potential functions for the field $\vec{F}$.

*Solution*:

$$\begin{aligned} f(x,y,z) &= \int y^3 - 2xz\, dx = xy^3 - x^2z + C_1(y,z) \\ f(x,y,z) &= \int 3xy^2 + 2z\, dy = xy^3 + 2yz + C_2(x,z) \\ f(x,y,z) &= \int 2y - x^2\, dz = 2yz - x^2z + C_3(x,y) \\ \Rightarrow\quad & \boxed{f(x,y,z) = xy^3 - x^2z + 2yz + C} \end{aligned}$$

(c) Use the Fundamental Theorem of Line Integrals to compute the work done on the particle.

*Solution*: Since $\vec{r}(0) = \langle 1,1,0\rangle$ and $\vec{r}(2) = \langle 8-6+1, 2, 1\rangle = \langle 3,2,1\rangle$ then

$$W = \int_C \vec{F}\cdot d\vec{r} = f(3,2,1) - f(1,1,0) = 3(8) - 9(1) + 2(2)(1) - 1 + 0 - 0 = \boxed{18}\,.$$

For extra credit, using the same initial and final points, find a simpler path between them and use it to compute the work again – directly this time.

*Solution*: We can use paths parallel to the axes and use the differential form:

- $C_1$: $(1,1,0) \to (3,1,0)$ then

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} M(x,1,0)\,dx + \cancelto{0}{N(x,1,0)\,dy} + \cancelto{0}{P(x,1,0)\,dz} = \int_1^3 1 - 2x(0)\,dx = \int_1^3 dx = 2;$$

- $C_2$: $(3,1,0) \to (3,2,0)$ then

$$\begin{aligned} \int_{C_2} \vec{F}\cdot d\vec{r} &= \int_{C_2} \cancelto{0}{M(3,y,0)\,dx} + N(3,y,0)\,dy + \cancelto{0}{P(3,y,0)\,dz} \\ &= \int_1^2 3(3)y^2 + 2(0)\,dy = \int_1^2 9y^2\,dx = \left[3y^3\right]_1^2 = 24 - 3 = 21; \end{aligned}$$

- $C_3$: $(3,2,0) \to (3,2,1)$ then

$$\begin{aligned} \int_{C_3} \vec{F}\cdot d\vec{r} &= \int_{C_3} \cancelto{0}{M(3,2,z)\,dx} + \cancelto{0}{N(3,2,z)\,dy} + P(3,2,z)\,dz \\ &= \int_0^1 2(2) - (3)^2\,dx = \int_0^1 -5\,dx = -5. \end{aligned}$$

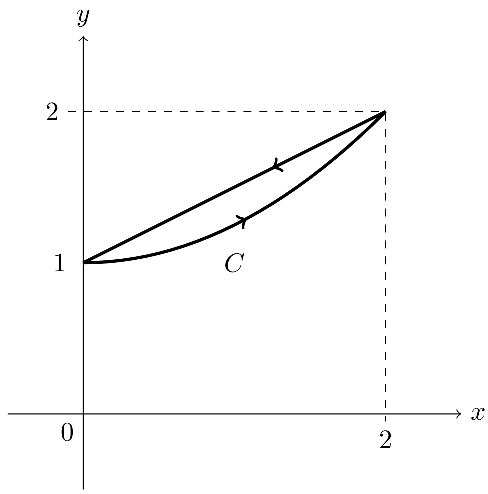
So putting it all together,

$$W = \int_C \vec{F}\cdot d\vec{r} = 2 + 21 - 5 = \boxed{18}\,. \quad \checkmark$$

**10.** Use Green's Theorem to evaluate

$$\oint_C \big(\sin(x^2) + 3y\big)\, dx + (\ln y + 4x)\, dy$$

where $C$ is the closed curve composed of the graph of $y = \frac{x^2}{4} + 1$ for $0 \le x \le 2$ followed by the line segment going from $(2, 2)$ to $(0, 1)$ as illustrated below:




*Solution*: The piecewise smooth closed simple curve $C$ is oriented positively and the line segment has equation $y = \frac{x}{2} + 1$. Setting $M = \sin\left(x^2\right) + 3y$ and $N = \ln y + 4x$, we have $M_y = 3$ and $N_x = 4$ and therefore by Green's theorem:

$$\oint_C M\ dx + N\ dy = \iint_R (N_x - M_y)\ dA = \int_0^2 \int_{\frac{x^2}{4}+1}^{\frac{x}{2}+1} (4-3)\ dy\ dx$$
$$= \int_0^2 \frac{x}{2} - \frac{x^2}{4}\ dx = \frac{x^2}{4} - \frac{x^3}{12}\bigg|_0^2 = 1 - \frac{8}{12} = \frac{4}{12} = \boxed{\frac{1}{3}}.$$