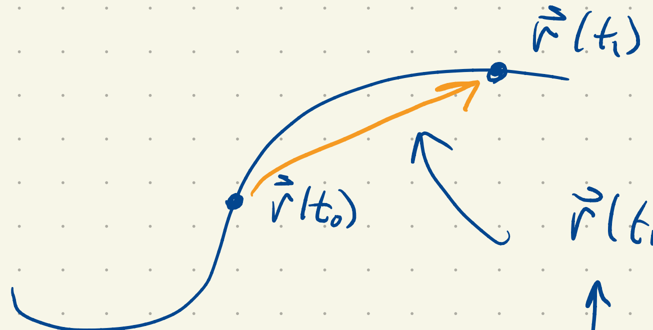
Section 13.2

Derivatives, Integrals of "vector valued functions"

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

"space curve"

If you want, think of it as position $r(t)$ in 3-d space as a function of time.



$\vec{r}(t_1) - \vec{r}(t_0)$ (units of m, e.g.)

Displacement vector, change in position

Change in time: $t_1 - t_0$

Average velocity: $$\frac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0}$$

We define $$\vec{r}\,'(t_0) = \lim_{t_1 \to t_0} \frac{\vec{r}(t_1) - r(t_0)}{t_1 - t_0}$$

What do I mean by this limit? Just apply component wise

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{r}\,'(t_0) = \lim_{t_1 \to t_0} \left\langle \frac{x(t_1) - x(t_0)}{t_1 - t_0}, \frac{y(t_1) - y(t_0)}{t_1 - t_0}, \frac{z(t_1) - z(t_0)}{t_1 - t_0} \right\rangle$$

$$= \left\langle \lim_{t_1 \to t_0} \frac{x(t_1) - x(t_0)}{t_1 - t_0}, \text{——}, \text{——} \right\rangle$$

$$= \langle x'(t_0), y'(t_0), z'(t_0) \rangle$$

You already know how to compute 1-d derivatives!

e.g. $\vec{r}(t) = \cos(\omega t)\,\hat{\imath} + \sin(\omega t)\,\hat{\jmath}$

$$|\vec{r}(t)| = 1$$

$$\vec{r}(0) = \hat{\imath}$$

$$\vec{r}\left(\frac{2\pi}{\omega}\right) = \hat{\imath} \quad \text{also.} \quad \left(\text{period } \frac{2\pi}{\omega}\right)$$

Before computing derivatives, if

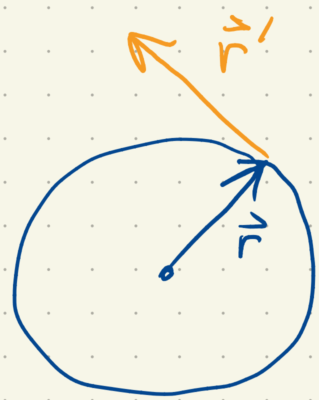
we complete one of length 2π in

time $\frac{2\pi}{\omega}$, what speed do we expect? ω

($\omega > 0$ anyway)

$$\vec{r}\,'(t) = -\omega \sin(\omega t)\,\hat{\imath} + \omega \cos(\omega t)\,\hat{\jmath}$$

Note: $\vec{r} \cdot \vec{r}\,' = -\omega \cos(\omega t)\sin(\omega t) + \omega \sin(\omega t)\cos(\omega t) = 0$



think of this as velocity.

How long is it anyway?

$$\|\vec{r}\,'\|^2 = \omega^2 \sin^2(\omega t) + \omega^2 \cos^2(\omega t) = \omega^2$$

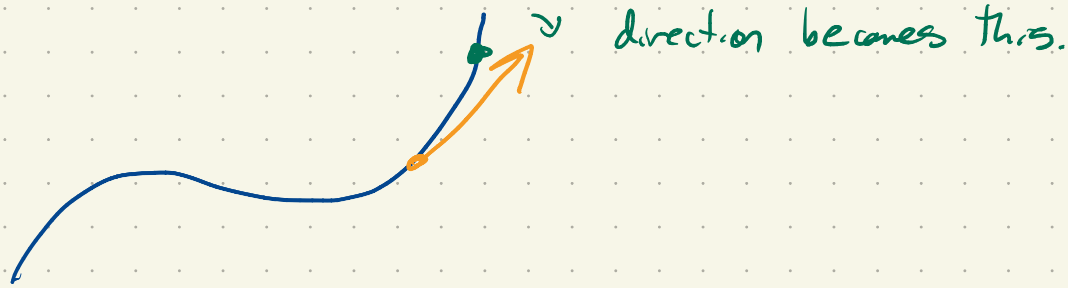
$$\|\vec{r}\,'\| = |\omega|$$

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

$$\vec{r}\,'(t) = 0 + \vec{v} = \vec{v}$$

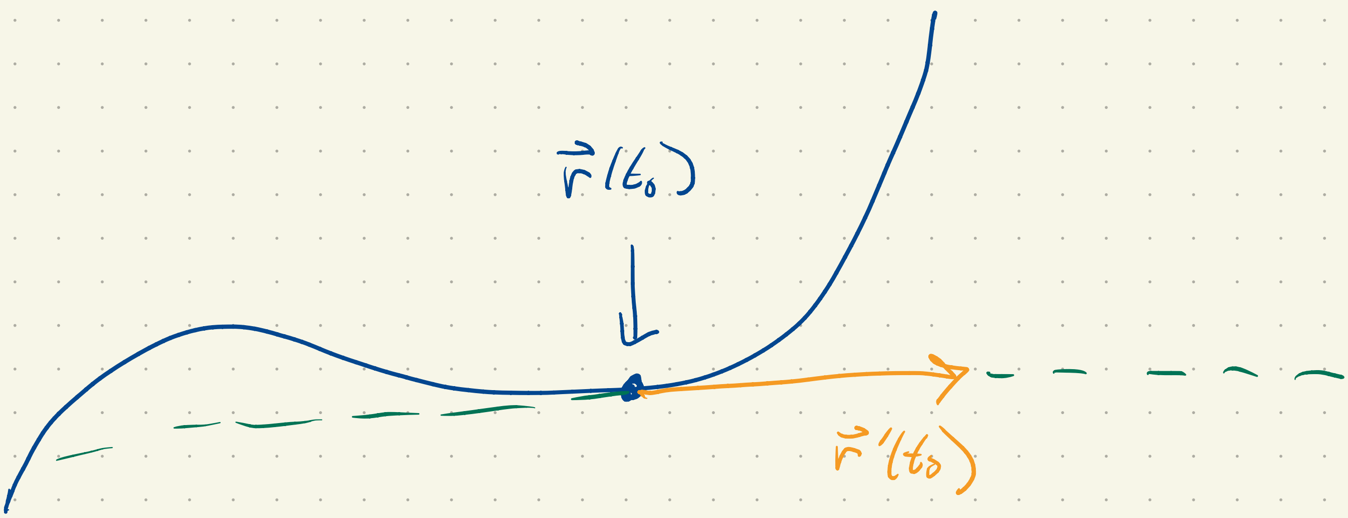
same velocity at all t.

On a general curve, we visualize $\vec{r}'$ pointing in a direction tangent to the curve



The length of $\vec{r}'$ is the speed of traversal.

Related to this direction is a notion of linear approx.

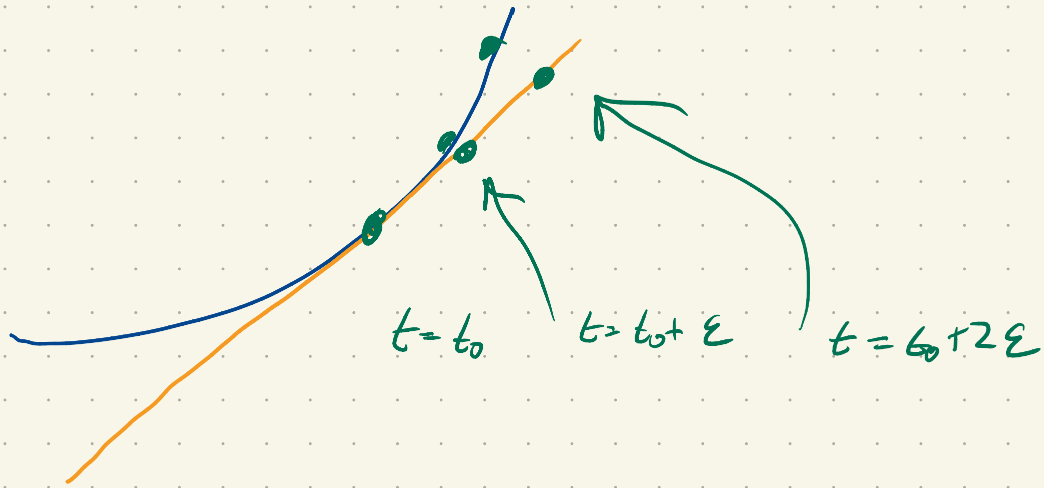


$$\vec{\ell}(t) = \vec{r}(t_0) + \vec{r}\,'(t_0)\,t$$

This is a constant velocity curve,

$$\vec{\ell}(t_0) = \vec{r}(t_0)$$
$$\vec{\ell}\,'(t_0) = \vec{r}\,'(t_0)$$

I'll call $\vec{\ell}(t)$ the tangent line to the curve, but it's really a parameterized line

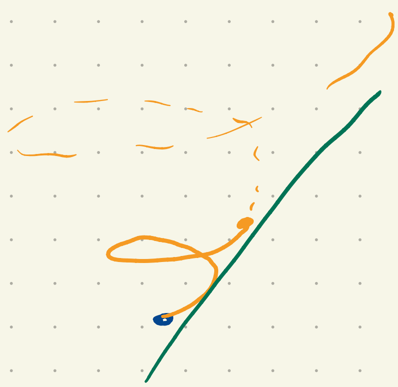


E.g. find the parametric ~~time~~ eq of the tangent line of

$$\vec{r}(t) = \langle t\cos t,\ t\sin t,\ t\rangle \quad \underline{\text{at}} \quad t=2\pi$$

$x^2+y^2=z^2$

lives on a cone



$$\vec{r}\,'(t) = \langle \cos t - t\sin t,\ \sin t + t\cos t,\ 1\rangle$$

$$\vec{r}\,'(2\pi) = \langle 1,\ 2\pi,\ 1\rangle$$

$$\vec{r}(2\pi) = \langle 2\pi,\ 0,\ 2\pi\rangle$$

$$\vec{l}(t) = \langle 2\pi, 0, 2\pi \rangle + (t-2\pi)\langle 1, 2\pi, 1 \rangle$$

$$= \langle t, \; 2\pi(t-2\pi), \; t \rangle$$

Cautionary tale:

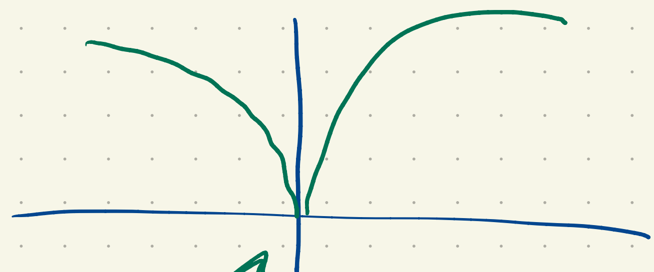
$$\vec{r}(t) = \langle t^3, t^2 \rangle$$

$\to y \geq 0$

x increases as t increases

$$y^3 = x^2$$

$$y = x^{2/3}$$



what's the deal?

t^3, t^2 look smooth!

If r' exists <u>and</u> $\neq 0$ we'll say the curve is smooth

Some rules

$$\frac{d}{dt}\left(\vec{r}(t)+\vec{s}(t)\right) = r'(t) + s'(t)$$

$$\frac{d}{dt}\, \vec{r}(t) \times \vec{s}(t) = \vec{r}'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}'(t)$$

$$\frac{d}{dt}\, \vec{r}(t) \cdot \vec{s}(t) = \vec{r}'(t) \cdot s(t) + \vec{r}(t) \cdot \vec{s}'(t)$$

$$\frac{d}{dt}\, \vec{r}(f(t)) = \vec{r}'(t)\, f'(t)$$

Suppose $|\vec{r}(t)| = 1$ for all t.

Then $\vec{r}(t) \cdot \vec{r}'(t) = 0$.

$$\frac{d}{dt}\, \vec{r}(t) \cdot \vec{r}(t) = \frac{d}{dt} 1 = 0$$

$$\vec{r}'(t) \cdot \vec{r}(t) + r(t) \cdot \vec{r}'(t) = 0$$

$$2\vec{r}(t) \cdot \vec{r}'(t) = 0 \quad \checkmark$$

Can replace 1 with any radius

Integration

$$\int_a^b \vec{r}(t)\,dt := \left\langle \int_a^b x(t)\,dt,\ \int_a^b y(t)\,dt,\ \int_a^b z(t)\,dt \right\rangle$$

Why would you do this?

$$\int_0^2 x^3 = \frac{x^4}{4}\Big|_0^2 = \frac{2^4}{4} - \frac{0^4}{4} = \frac{16}{4} = 4.$$

What happened here

$$f(x) = x^3$$

$$F(x) = \frac{x^4}{4} \quad , \ F'(x) = f(x)$$

$$\int_0^2 F'(x)\,dx = F(2) - F(0)$$

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \text{FTC!}$$

integrate a rate of change and you get a net change.

$$\int_a^b \vec{r}\,'(t)\,dt = \left\langle \int_a^b x'(t)\,dt,\ \int_a^b y'(t)\,dt,\ \int_a^b z'(t)\,dt \right\rangle$$

$$= \langle x(b)-x(a),\ y(b)-y(a),\ z(b)-z(a) \rangle$$

$$= \vec{r}(b) - \vec{r}(a) \qquad \text{(FTC)}$$

Same principle: integrate a velocity and you get a net change in position. (displacement)

We'll sometimes write $\vec{r}\,'(t) = \vec{v}(t)$

$$\int_a^b \vec{v}(t) = \vec{r}(b) - \vec{r}(a)$$

E.g. Suppose $r'(t) = \vec{v}(t) = 5\hat{\imath} + (2 - 9.8t)\hat{\jmath}$.

and $\vec{r}(0) = \langle 1, 3, 2 \rangle$

Job: Find $\vec{r}(t)$

Method 1:

$$\int_0^t \vec{v}(t) = \int_0^t \vec{r}'(s)\,ds = \vec{r}(t) - \vec{r}(0)$$

$$\int_0^2 \langle 5, 2-9.8t, 0\rangle dt = \vec{r}(t) - \langle 1, 3, 2\rangle$$

$$\langle 5s\Big|_0^t, 2s - \frac{9.8}{2}s^2\Big|_0^t, 0\rangle = \vec{r}(t) - \langle 1, 3, 2\rangle$$

$$\langle 5t, 2t - \frac{9.8}{2}t^2, 0\rangle = \vec{r}(t) - \langle 1, 3, 2\rangle$$

$$\vec{r}(t) = \langle 1+5t, 3+2t-\frac{9.8}{2}t^2, 2\rangle$$

Method 2:

We hue a notion of indefinite integrals as well

constant vector;

$$\int r'(t)\,dt = \vec{r}(t) + \vec{c}$$

$$\vec{r}(t) = \int \langle 5,\ 2 - 9.8t,\ 0 \rangle\, dt + \vec{c}$$

$$= \langle 5t,\ 2t - \tfrac{9.8}{2}t^2,\ 0 \rangle$$

$$\vec{r}(t) = \langle 5t,\ 2t - \tfrac{9.8}{2}t^2,\ 0 \rangle + \vec{c}$$

$t = 0$

$$\langle 1, 3, 2 \rangle = \langle 0, 0, 0 \rangle + \vec{c}$$

$$\vec{c} = \langle +1, +3, +2 \rangle$$

$$\vec{r}(t) = \langle 5t + 1,\ 2t - \tfrac{9.8}{2}t^2 + 3,\ +2 \rangle$$