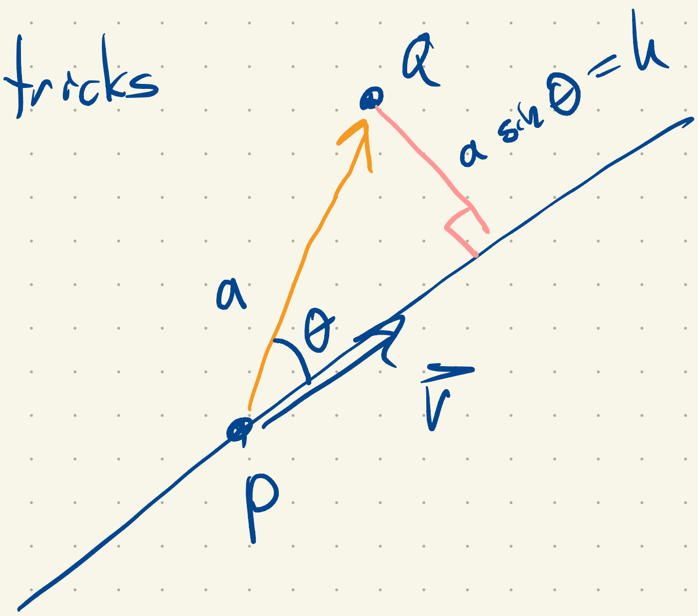
Fun tricks



How far is $\vec{q}$ from the line? (h)

It's $a \sin\theta$. But $\|\overrightarrow{PQ} \times \vec{v}\| = a\|\vec{v}\|\sin\theta$

So $h = a\sin\theta = \dfrac{\|\overrightarrow{PQ} \times \vec{v}\|}{\|\vec{v}\|}$

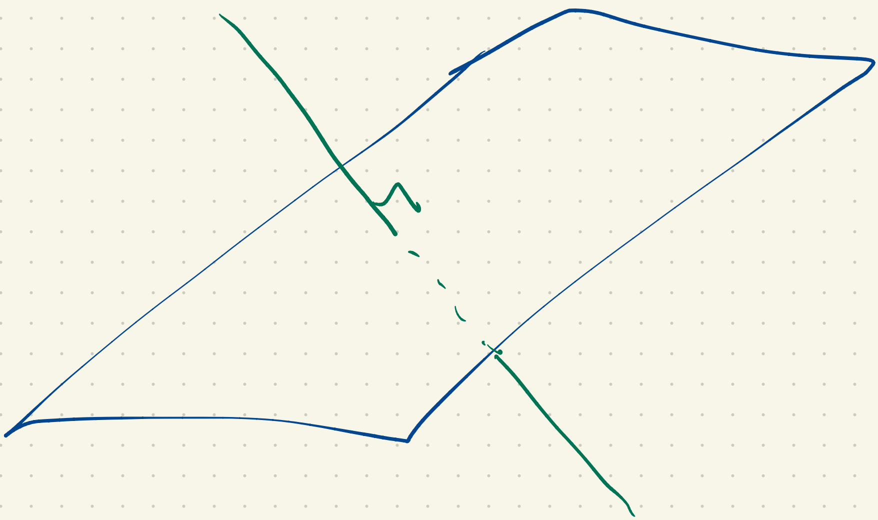
$= \left\| \overrightarrow{PQ} \times \left(\dfrac{\vec{v}}{\|\vec{v}\|} \right) \right\|$

unit vector!

Equations of planes.

What data do you need to describe a plane?

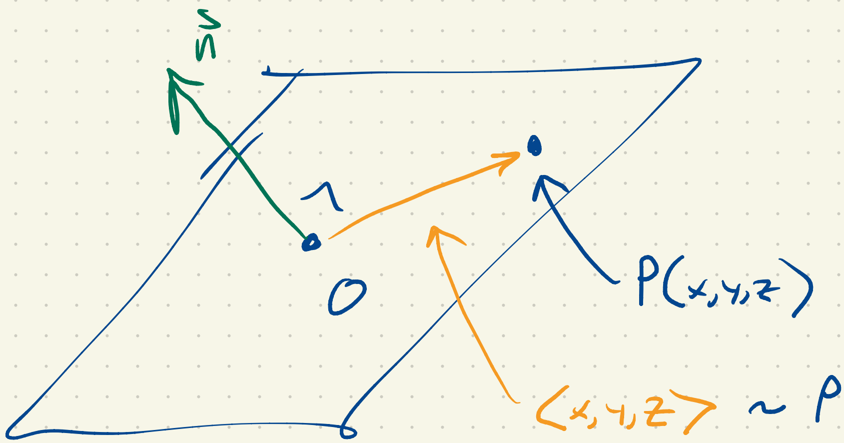
In 3-dimensions every plane has a unique orthogonal direction



A vector pointing in the orthogonal direction is known as a <u>normal</u> <u>vector</u>

Special case:

What if the plane passes through the origin?



$$\vec{n} \cdot \langle x,y,z\rangle = 0$$

$$\vec{n} = \langle a,b,c\rangle$$

$$ax + by + cz = 0$$

this is the equation of a plane through origin with normal $\langle a,b,c\rangle$

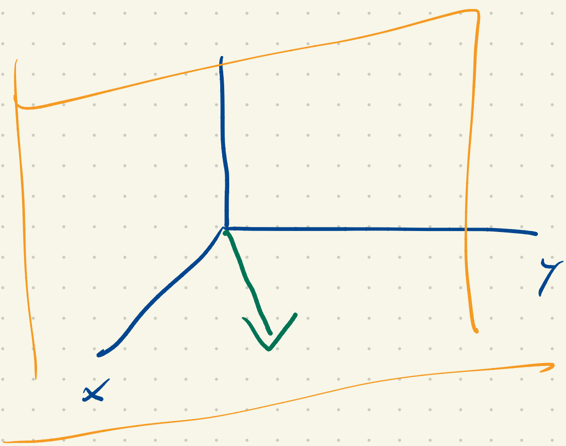
What happens if you replace $\vec{n}$ with $7\vec{n}$?

Plane doesn't change.

Formula becomes $7ax + 7by + 7cz = 0$

and the same points satisfy this relation.

e.g. Given $\vec{n} = \langle 1, 1, 0 \rangle$ is



P(3, 2, 6)

on the plane thru O

w/ this normal?

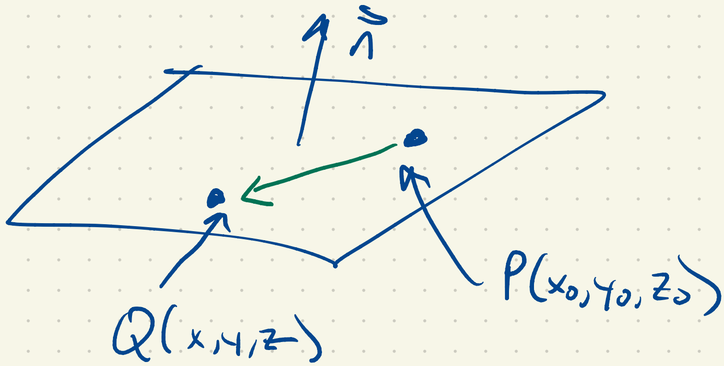
Is Q(−1, 1, 5)?

$$x + y = 0$$

$3 + 2 \neq 0$ P, no

$-1 + 1 = 0$ Q, yes

But not every plane is thru the origin



$$\vec{V} = \langle x - x_0,\ y - y_0,\ z - z_0 \rangle$$

$\vec{n} \cdot \vec{V} = 0$ again

$\vec{n} = \langle a, b, c \rangle$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$\langle a, b, c \rangle$ normal

$P(x_0, y_0, z_0)$ some spot on plane.

We can also write this as

$$ax+by+cz = \underbrace{ax_0+by_0+cz_0}_{d}$$

$$ax+by+cz=d$$

$\langle a,b,c\rangle$ which normal

d label for the plane w/ this normal

less informative

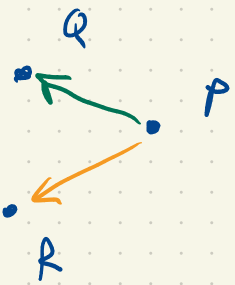
$d=0 \iff$ is plane thru 0.

E.g. Find the equation of the plane thru the three points

$P(1,0,2)$

$Q(-1,3,4)$

$R(3,5,7)$



$\overrightarrow{PQ} = \langle -2, 3, 2 \rangle$

$\overrightarrow{PR} = \langle 2, 5, 5 \rangle$

Now we need a vector ortho to both.

to get the normal.

$\overrightarrow{PQ} \times \overrightarrow{PR} = \vec{n}$ will do!

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & 2 \\ 2 & 5 & 5 \end{vmatrix} = \hat{i}(5) + \hat{j}\, 14 + \hat{k}(-6)$$

$$= \langle 5, 14, -6 \rangle$$

$$\vec{n} \cdot (\langle x, y, z \rangle - \vec{P}) = 0$$

$\uparrow$ or $\vec{Q}$ or $\vec{R}$

$$\vec{n} \cdot (\langle x-1, y, z-2 \rangle) = 0$$

$$5(x-1) + 14y - 6(z-2) = 0$$

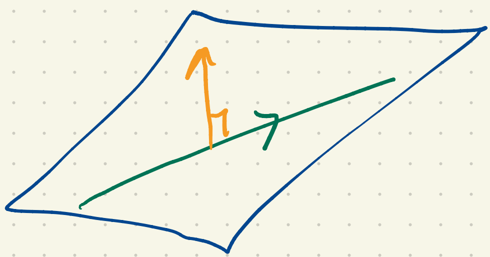
On $5x + 14y - 6z = 5 + 12$
$= 17.$

E.g. Find line of intersection between

$x + y + z = 1$ $\quad$ $x - 2y + 3z = 1$

$\uparrow$

$\vec{n}_1 = \langle 1, 1, 1 \rangle$ $\quad$ $\vec{n}_2 = \langle 1, -2, 3 \rangle$



If $\vec{v}$ is in the direction of a line in the plane then $\vec{v} \cdot \vec{n} = 0$.

So this line in the intersection has to have a direction $\perp$ to $\vec{n}_1$ and $\perp$ to $\vec{n}_2$.

How 'bout that cross product

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 3+2, -(3-1), -2+1 \rangle$$

$$= \langle 5, -2, -1 \rangle$$

Great! Now we know a normal. We just need a point.

Let's pick the point on the line where $x = 0$

At that point $\quad y + z = 1 \quad \Rightarrow \quad 2y + 2z = 2$

$\qquad\qquad\qquad -2y + 3z = 1 \qquad\quad -2y + 3z = 1$

$$5z = 3$$

$$z = 3/5$$

$$y = 2/5$$

Point: $\langle 0, \frac{2}{5}, \frac{3}{5} \rangle$

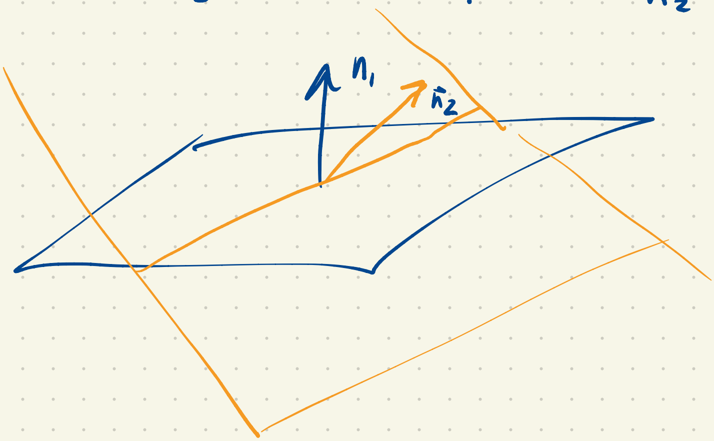
$$\vec{r} = \langle 0, \tfrac{2}{5}, \tfrac{3}{5} \rangle + t \langle 5, -2, -1 \rangle$$

$$x = 5t, \quad y = \frac{2}{5} - 2t \quad z = \frac{3}{5} - t$$

E.g. Angle between planes

$x + y - z = 5$ $\quad\quad$ $\vec{n}_1 = \langle 1, 1, -1 \rangle$

$3x - y + 4z = 9$ $\quad\quad$ $\vec{n}_2 = \langle 3, -1, 4 \rangle$



Angle between planes is just angle between normals

$$\vec{n}_1 \cdot \vec{n}_2 = \|n_1\| \, \|n_2\| \cos\theta$$

$$3 - 1 - 4 = -2$$

$$\|\vec{n}_1\| = \sqrt{3}$$

$$\|\vec{n}_2\| = \sqrt{9 + 1 + 16} = \sqrt{26}$$

$$\theta = \arccos\left(\frac{-2}{\sqrt{3}\sqrt{26}}\right) =$$