Dot Products + Physics

If a constant force $\vec{F}$ is applied to a body

that moves from P to Q

then the body gains/loses energy.

This change is the <u>work</u> done on the body.

work is a change in energy, a scalar

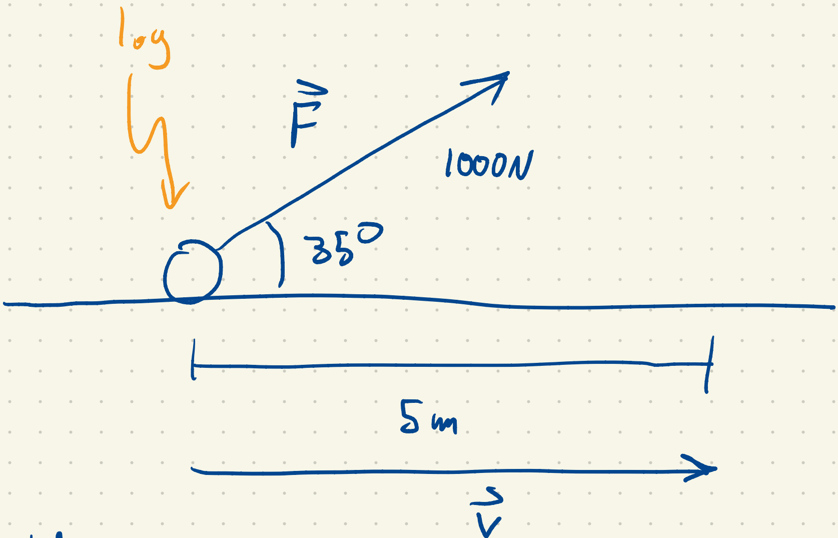
$$\text{work} = \underbrace{\vec{F}}_{kg\frac{m}{s^2} = N} \cdot \underbrace{\overrightarrow{PQ}}_{m}$$

ftlb also

:(

$$[\text{work}] = kg\ m^2/s^2 = J, \quad \text{Joule}$$

unit of energy

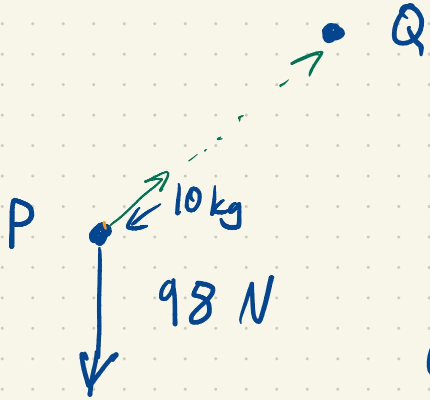


Work done

$\vec{F} \cdot \vec{v}$ $\quad\quad$ $\vec{v} = 5\hat{i}$ m

$$\vec{F} = 1000\left(\cos 35^\circ\,\hat{i} + \sin 35^\circ\,\hat{j}\right) N$$

$$\vec{F} \cdot \vec{v} = 5000 \cos 35^\circ = 4095.76 \text{ J}$$



P (1m, 3m)

Q (4m, 7m)

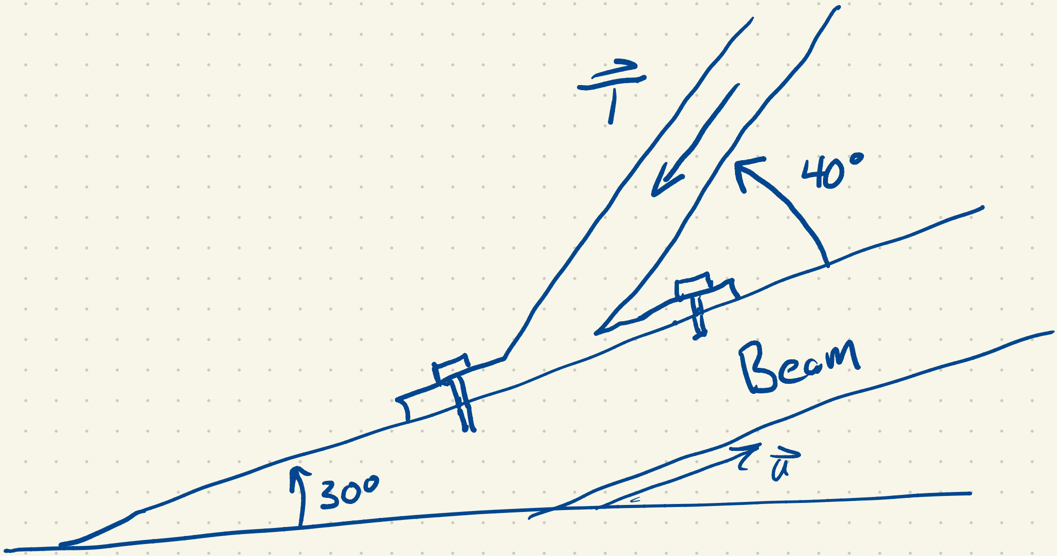
$$\vec{PQ} = \langle 3, 4 \rangle$$

$$\vec{F} = \langle 0, -98 \rangle$$

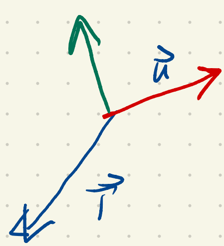
$$\vec{F} \cdot \vec{PQ} = -392\ J$$

Gravity robs it of 400 J of energy to move from $\vec{P}$ to $\vec{Q}$.

Orthogonal Projection

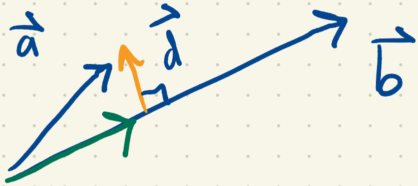


Question: How much of $\vec{T}$ is parallel to the large beam? We care, because shear forces matter at the bolts.



We can write $\vec{T}$ as a sum of pieces parallel to and perp to $\vec{u}$

(Orthogonal) Projection



projection of $\vec{a}$ along $\vec{b}$

(write $\vec{a}$ as sum green + $\perp$ to $\vec{b}$)

$$\vec{a} = c\vec{b} + \vec{d} \qquad \vec{d} \perp \vec{b}$$

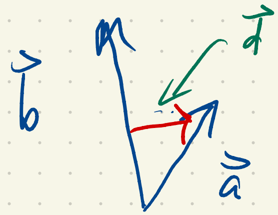
$$\vec{a} \cdot \vec{b} = c\,|\vec{b}|^2$$

$$c = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}$$

$$\vec{a} = \underbrace{\boxed{\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\,\vec{b}}}_{\operatorname{proj}_{\vec{b}} \vec{a}} + \underbrace{\vec{d}}_{\vec{a} - \operatorname{proj}_{\vec{b}} a}$$

e.g. $\vec{a} = \langle 1, 2 \rangle$

$b = \langle -1, 4 \rangle$



$$\text{proj}_{\vec{b}}\, \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b}$$

$$\vec{a} \cdot \vec{b} = 1 \cdot (-1) + 2 \cdot 4 = 7$$

$$\|\vec{b}\|^2 = 1 + 16 = 17$$

$$\text{proj}_{\vec{b}}\, \vec{a} = \frac{7}{17} \langle -1, 4 \rangle = \left\langle -\frac{7}{17}, \frac{28}{17} \right\rangle$$

$$= \langle 0.41, 1.64 \rangle$$

$$d = \vec{a} - \text{proj}_{\vec{b}}\, a$$

$$= \langle 1, 2 \rangle - \langle 0.41, 1.64 \rangle$$

$$= \langle 0.59, 0.35 \rangle$$

Rules:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} + \vec{u} \cdot (c\vec{v})$$

Cross Product

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

$\vec{a} \times \vec{b}$

First how to compute.

Then what it is geometrically

Then application.

Then, maybe, determinants

Result is a <u>vector</u>.

$$\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{array}$$

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\hat{i}$$

$$- (a_1 b_3 - a_3 b_1)\hat{j}$$

$$+ (a_1 b_2 - a_2 b_1)\hat{k}$$

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{array}$$

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{array}$$

What a mess! What could this be good for...

Some properties

1) $\vec{a} \times \vec{a} = \vec{0}$

$(a_2 a_3 - a_3 a_2)\hat{\imath} - (a_1 a_3 - a_3 a_1)\hat{\jmath}$

etc....

2) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(anticommutative)

$$\left.\begin{matrix} a_2 b_3 - a_3 b_2 \\ b_2 a_3 - b_3 a_2 \end{matrix}\right] \text{etc.}$$

3) $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ (!)

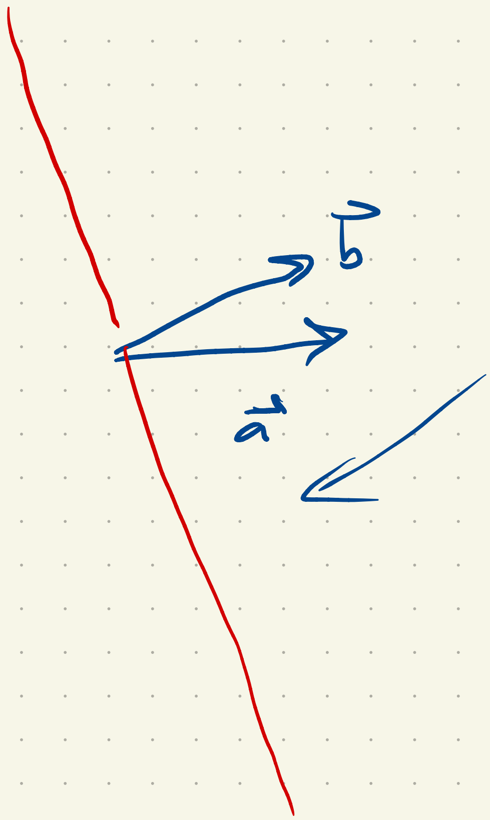
$$a_1\left(\cancel{a_2 b_3} - \cancel{a_3 b_2}\right) - a_2\left(\cancel{a_1 b_3} - \cancel{a_3 b_1}\right) + a_3\left(\cancel{a_1 b_2} - \cancel{a_2 b_1}\right)$$

4) $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$

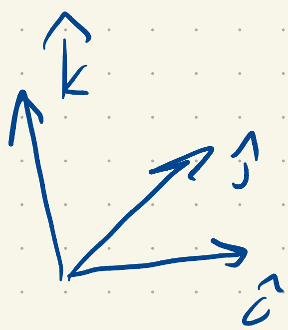
$$\vec{b} \cdot (\vec{a} \times \vec{b}) = -\vec{b}(\vec{b} \times \vec{a})$$
$$= -0 = 0$$

Whoa! Whatever this cross product thing is

it is perpendicular to both $\vec{a}$ and $\vec{b}$!



It points somewhere along this line!



a) How long?

b) which side?

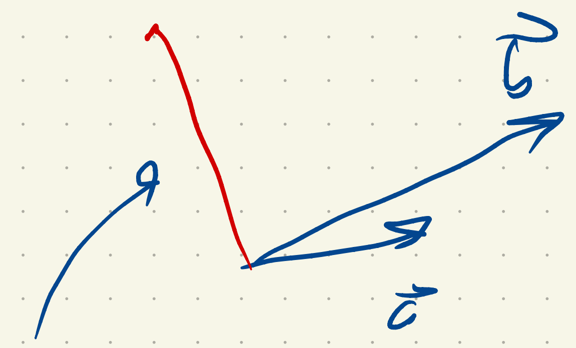
$\hat{i}\ \hat{j}\ \hat{k}\ \hat{i}\ \hat{j}$

$$\hat{i} \times \hat{j} = \hat{k} \qquad \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \qquad 0\hat{i} + 0\hat{j} + 1\hat{k}$$

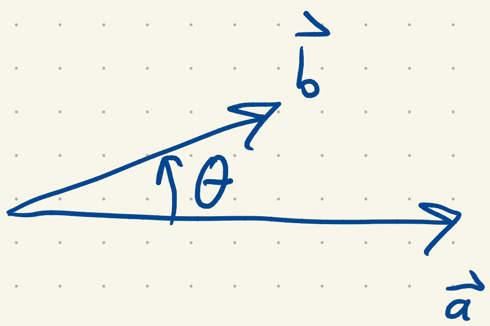
$$\left.\begin{array}{l} \hat{j} \times \hat{k} = \hat{i} \\ \hat{k} \times \hat{i} = \hat{j} \end{array}\right] \text{exercise}$$

Result is always right handed

always true!



$\vec{a} \times \vec{b}$ along here. How much?



$$\vec{b} = \|\vec{b}\|\cos\theta\,\hat{c} + \|\vec{b}\|\sin\theta\,\hat{j}$$

$$\vec{a} = \|\vec{a}\|\,\hat{c}$$

$$\vec{a} \times \vec{b} = \|\vec{a}\|\,\|\vec{b}\|\sin\theta\,\hat{k}$$

↑
unit

$(0 \le \theta \le 180^\circ)$

$0 \le \sin\theta \le 1$

$$\boxed{\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta}$$