Please see the rules below.

1. Let $C(\mathbb{R})$ denote the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $C(\mathbb{R})$ is dense in $\mathbb{R}^{\mathbb{R}}$ in the product topology, but not dense in the box topology.

2. The Möbius band is the quotient of $[0,1] \times \mathbb{R}$ where $(0,y) \sim (1,-y)$.

a) Show that the Möbius band is a 2-manifold.

b) Show that the Möbius band is homotopy equivalent to a circle.

c) No rigor please, just a picture or two: what familiar space is the 1-point compactificaiton of the Möbius band?

3. You will recall that showing that $\mathbb{RP}^n$ is Hausdorff was a hassle. Here's a slick technique when the quotient you are looking at arises (as it often does!) from a group action.

a) Lee 3-22 b

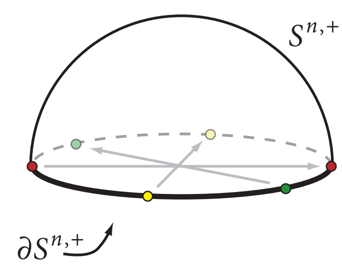
b) Show $\mathbb{RP}^n$ is Hausdorff. (Hurrah Closed Map Theorem!)

4. Suppose the Earth is a sphere and that temperature is a continuous function of position on Earth. Prove that there is a point p on Earth where the temperature is the same as at its antipodal point $-p$.

5.

a) Show that the upper half sphere $S^{n,+}$ with antipodal points on $\partial S^{n,+}$ identified is homeomorphic to $\mathbb{RP}^n$.

b) Consider $X = \{(xy, yz, zx, x^2, y^2, z^2) \in \mathbb{R}^6 : x^2 + y^2 + z^2 = 1\}$. Prove that this set is homeomorphic to $\mathbb{RP}^2$.



Gluing a half sphere.

6. Suppose G is a topological group, A is a closed subset, and B is a compact subset. Show that AB is closed. Hint: Nets!

7. Recall that a set $A \subset X$ is a retract of X if there is a continuous $f : X \to A$ such that $f(a) = a$ for all $a \in A$.

a) Show that if X is Hausdorff and A is a retract of X then A is closed.

b) Let A be a two point subset of $\mathbb{R}^2$. Show that it is not a retract of $\mathbb{R}^2$.

c) Show that the closed ball $\overline{\mathbb{B}^2}$ is a retract of $\mathbb{R}^2$.

d) Show that S^1 is not a retract of $\mathbb{R}^2$.

8. Consider a map $g : S^1 \to S^1$. Using degree theory techniques, show that g admits a lift $\tilde{g} : S^1 \to \mathbb{R}$ if and only if $\deg(g) = 0$.

9. Lee 9-5

10. Compute the fundamental groups of the following spaces:

a) The projective plane with two points removed.

b) A torus with two points removed.

c) The union of the unit sphere S^2 in $\mathbb{R}^3$ with the z-axis.

Rules and format:

- You are welcome to discuss this exam with me (David Maxwell) to ask for hints and so forth.
- If you find a suspected typo, please contact me as soon as possible and I will communicate it to the class if needed.
- You may not discuss the exam with anyone else (including an AI) until after the due date/time.
- You are permitted to reference either of the two texts used for the class (Lee or Crossley). No other resources are permitted.
- Each problem is weighted equally (10 points each).
- The due date/time is absolutely firm.
- We will schedule a hints sessions for this exam.