Goal: A space X is compact iff every net in X has a convergent subnet.

Two parts

a) A net has a subnet converging to x if and only if x is a cluster point of the net.

b) A space X is compact iff every net in X has a cluster point.

Prop: Suppose the net $\langle x_\alpha \rangle_{\alpha\in A}$ has a subnet $\langle x_{\alpha_\beta} \rangle_{\beta\in B}$ converging to some x. Then x is a cluster point.

Pf: Let $\langle x_{\alpha_\beta} \rangle_{\beta\in B}$ be a subnet converging to x.

Let U be an open set that contains x.

Let $\alpha_0 \in A$. (Job: show there exists $\alpha \geq \alpha_0$ such that $x_\alpha \in U$.)

Pick β_1 so that $\alpha_{\beta_1} \geq \alpha_0$. (cofinal)

Pick β_2 so that if $\beta \geq \beta_2$ then $x_{\alpha_\beta} \in U$. (convergence of subnet)

Pick β_3 with $\beta_3 \geq \beta_1$ and $\beta_3 \geq \beta_2$. (directedness)

I claim $\alpha_{\beta_3} \geq \alpha_0$ and $x_{\alpha_{\beta_3}} \in U$.

Observe $\underbrace{\alpha_{\beta_3} \geq \alpha_{\beta_1}}_{\text{(increasing)}} \overbrace{{} \geq \alpha_0}^{\text{from above}}$ and hence $\underbrace{\alpha_{\beta_3} \geq \alpha_0}_{\text{transitivity.}}$.

Moreover since $\beta_3 \geq \beta_2$, $x_{\alpha_{\beta_3}} \in U$ by the choice of β_2.

$\square$

Prop: Suppose $\langle x_\alpha \rangle_{\alpha \in A}$ is a net with a cluster point x. Then there exists a subnet of the net that converges to x.

Pf: Let $\mathcal{B} = \{(U, \alpha) : U \in \mathcal{U}(x)$ and $\alpha \in A$ and $x_\alpha \in U\}$.

We order $\mathcal{B}$ by $(U_1, \alpha_1) \geq (U_2, \alpha_2)$ if $U_1 \subseteq U_2$ and $\alpha_1 \geq \alpha_2$. I claim $\mathcal{B}$ is a directed set.

Then the ordering on $\mathcal{B}$ is evidently reflexive and transitive.

To show it is directed pick (U_1, α_1) and (U_2, α_2) in $\mathcal{B}$.

Let $U_3 = U_1 \cap U_2$. Using directedness we can find $\hat{\alpha}$ where $\hat{\alpha} \geq \alpha_1$ and $\hat{\alpha} \geq \alpha_2$. $(U_3, \hat{\alpha})$

Since x is a cluster point we can find $\alpha_3 \geq \hat{\alpha}$

and $x_{\alpha_3} \in U_3$. Then $(U_3, \alpha_3) \in B$.

Moreover $(U_3, \alpha_3) \geq (U_2, \hat{\alpha}) \geq (U_2, \alpha_2)$.

Similarly $(U_3, \alpha_3) \geq (U_1, \alpha_1)$.

We define a map $B \to A$ by $(U, \alpha) \to \alpha$.

To see this map is increasing suppose $(U_1, \alpha_1) \geq (U_2, \alpha_2)$.

Then $\alpha_1 \geq \alpha_2$ by def of the ordering on B.

The map is cofinal since for every $\alpha \in A$
$(X, \alpha) \in B$ (since X is open and contains x and $x_\alpha \in X$)

(I.e. the map from B to A is surjective and here cofinal)

$\beta = (W, \gamma)$

$x_{\alpha_\beta} = x_\gamma$

Consider the subnet $\langle x_{\alpha_\beta} \rangle_{\beta \in B}$.

I claim it converges to x.

Job: Show that given an open set U containing x there is a β_0 so that if $\beta \geq \beta_0$ $x_{\alpha_\beta} \in U$.

Let U be an open set containing x.

Since x is a cluster point there exists an α_0 with $x_{\alpha_0} \in U$. Let $U_0 = U$.

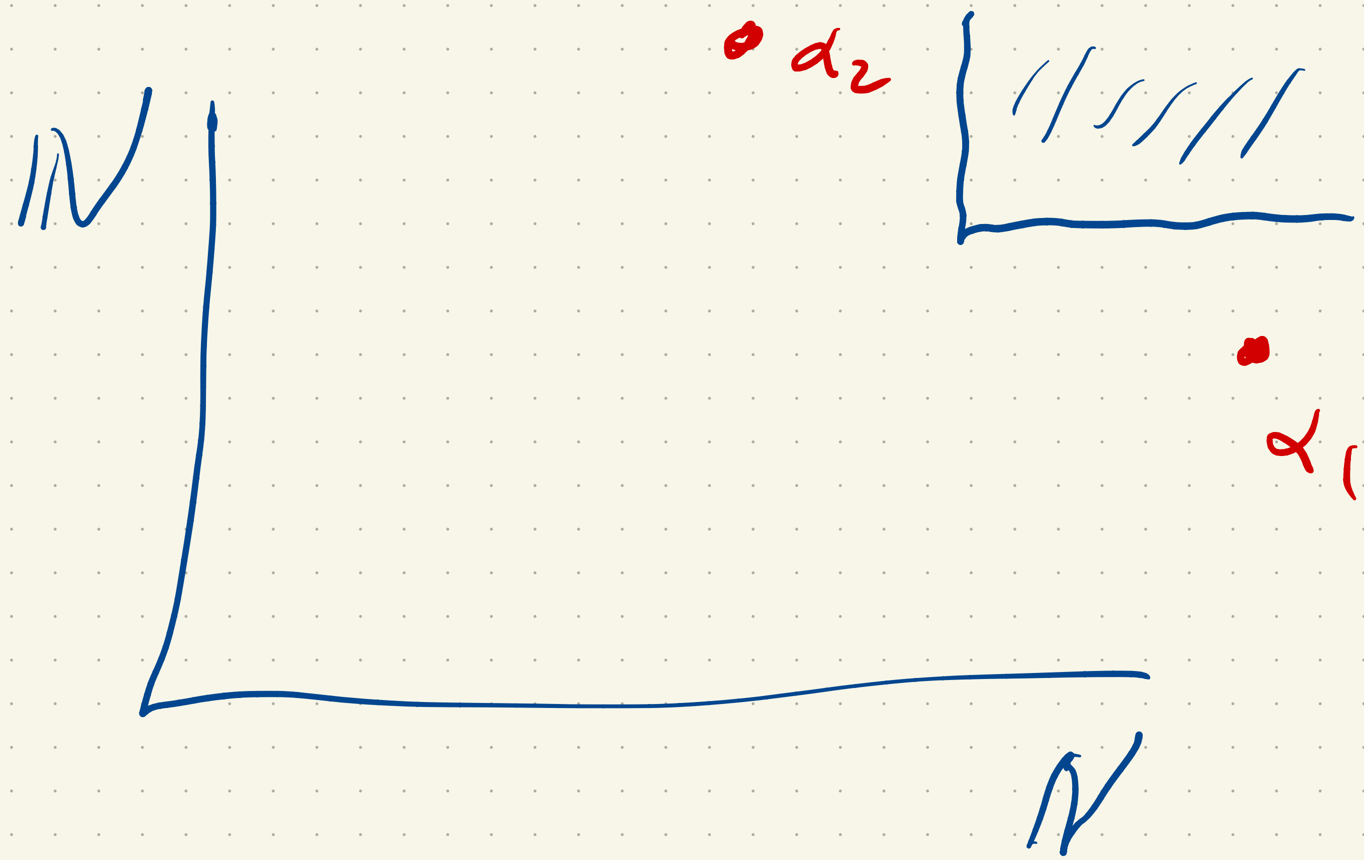
Let $\beta_0 = (U_0, \alpha_0)$.

Suppose $\beta = (V, \alpha) \in \mathcal{B}$ and $\beta \geq \beta_0$.

Job: Show $x_{\alpha_\beta} \in U$.

Observe $x_{\alpha_\beta} = x_\alpha$ and $x_\alpha \in V$ by definition of $\mathcal{B}$. Moreover, since $\beta \geq \beta_0$ $V \subseteq U_0 = U$.

So $x_\alpha \in U$ as required. $\square$


$\mathbb{N}$
α_2
α_1
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