$f : X \to Y$ is continuous iff

whenever $\langle x_\alpha \rangle_{\alpha \in A}$ is a net in X converging to some x,

then $\langle f(x_\alpha) \rangle_{\alpha \in A}$ converges to $f(x)$.

"f takes convergent nets to convergent nets"

Lemma: If $f : X \to Y$ is continuous then it takes convergent nets to convergent nets.

Pf: Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X converging to x.

Job: Show $\langle f(x_\alpha) \rangle_{\alpha \in A}$ converges to $f(x)$.

Let U be an open set in Y containing $f(x)$.

Job: Show a tail of $\langle f(x_\alpha) \rangle_{\alpha \in A}$ lies in U.

Let $W = f^{-1}(U)$. Since f is continuous, W is open in X.

Moreover, $x \in W$. Since $x_\alpha \to x$ there exists α_0 such that if $\alpha \geq \alpha_0$ then $x_\alpha \in W$.

Then if $\alpha \geq \alpha_0$ then $f(x_\alpha) \in f(W) \subseteq U$.

In particular, U contains a tail of the net $\langle f(x_\alpha) \rangle_{\alpha \in A}$

Lemma: Suppose $f: X \to Y$ such that whenever $\langle x_\alpha \rangle_{\alpha \in A}$ converges to x, $\langle f(x_\alpha) \rangle_{\alpha \in A}$ converges to $f(x)$.

Then f is continuous.

Pf: Suppose f has the assumed properties. To show f is continuous we'll show that the preimages of closed sets are closed.

Let $A \subseteq Y$ be a closed set. Let $x \in \overline{f^{-1}(A)}$.

Job: show $x \in f^{-1}(A)$

Since $x \in \overline{f^{-1}(A)}$ there is a net $\langle x_\beta \rangle_{\beta \in B}$ in $f^{-1}(A)$ converging to x. But then $f(x_\beta) \to f(x)$.

Since A is closed and since $\langle f(x_\beta) \rangle$ is a net in the closed set A

converging to $f(x)$, $f(x) \in A$.

Hence $x \in f^{-1}(A)$. $\square$

Next HW: You'll characterize Hausdorffness using nets.

X is Hausdorff iff convergent nets have unique limits

Today: We'll characterize compactness using nets.

"A set X is compact iff every net has a convergent subnet."

Subnets:

$f(\beta) = \alpha_\beta$

$\beta_1 \le \beta_2 \Rightarrow \alpha_{\beta_1} \le \alpha_{\beta_2}$

Let A and B be directed sets.

We say a map $f : B \to A$ is

- increasing if whenever $\beta_1 \le \beta_2$ in B, $f(\beta_1) \le f(\beta_2)$ in A.
- cofinal if for every $\alpha \in A$ there exists $\beta \in B$ such that $f(\beta) \geq \alpha$.

Def: Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net. A subnet of this net is a net of the form $\langle x_{f(\beta)} \rangle_{\beta \in B}$ where $f : B \to A$ is an increasing cofinal map between directed sets.

$A = \mathbb{N} \quad B = \mathbb{N}$

$f : B \to A$ increasing, cofinal → $f(B)$ not bounded above

$\not\Rightarrow$ strictly increasing

$1 \to 1 \quad 2 \to 1 \quad 3 \to 1 \quad 4 \to 4 \quad 5 \to 5, \ldots$

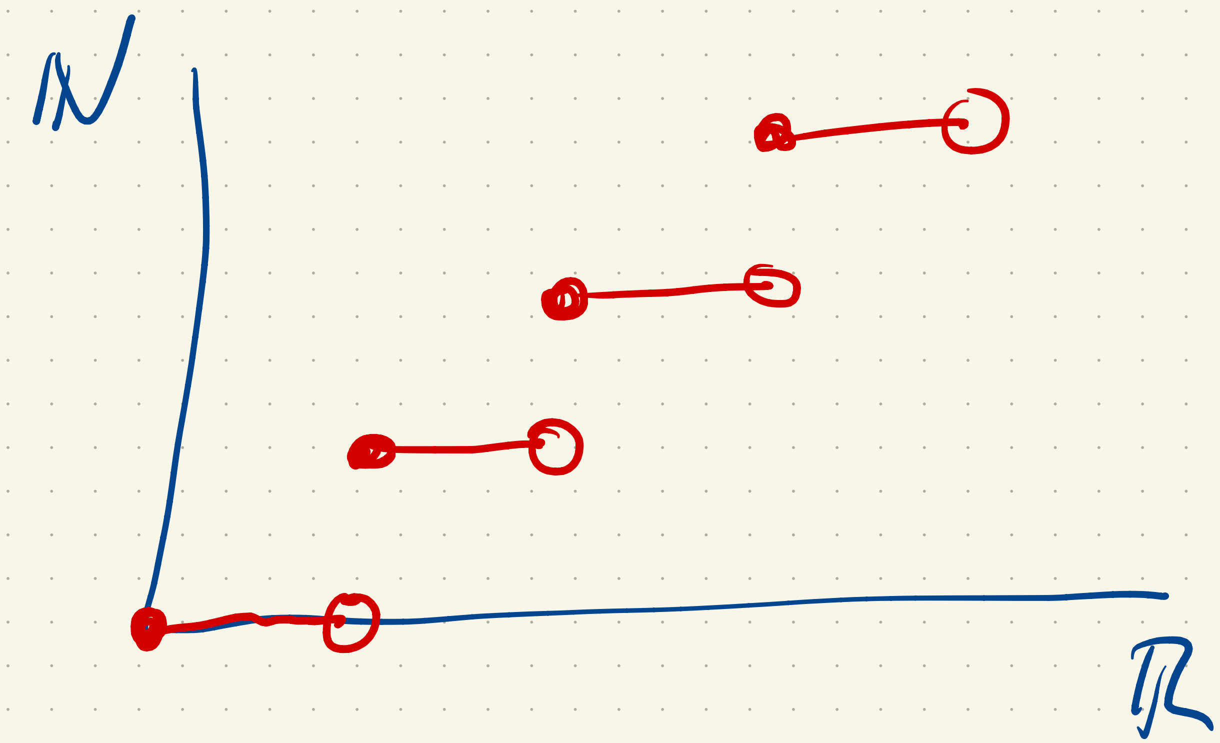
$\{x_n\} \quad \{x_{n_k}\} \quad n_k \quad n_1 < n_2 < n_3 < \cdots$

Subsequences are subnets but when $B = A = \mathbb{N}$ a subnet need not be a subsequence.

$B = \mathbb{R}_{\geq 0}$ $\quad A = \mathbb{N}$

$f : B \to A$

$f(\beta) = \lfloor \beta \rfloor$



$f(0) = 0$

$f(0.6) = 0$

$f(1) = 1$

$f(1.3) = 1$

$m \in \mathbb{N}$

$f(n) = n \geq m$ ✓

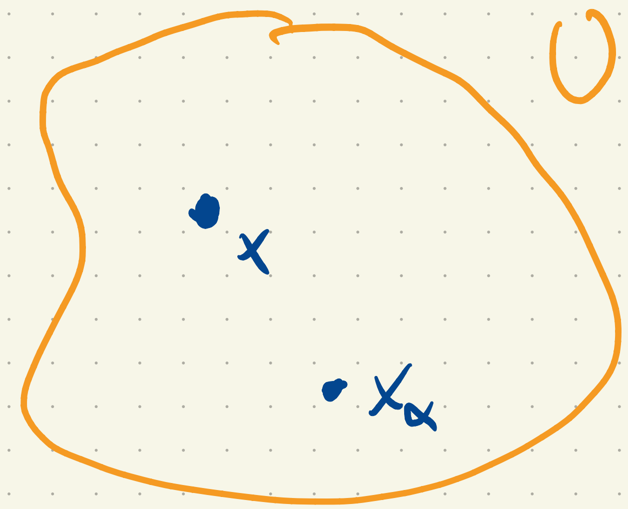
$\langle x_n \rangle_{n \in \mathbb{N}}$ $\quad \langle x_{f(z)} \rangle_{z \in \mathbb{R}_{\geq 0}}$

$f(\beta) = \alpha$ $\qquad\qquad$ $\langle x_\alpha \rangle_{\alpha \in A}$ $\qquad\qquad$ $\langle x_{\alpha_\beta} \rangle_{\beta \in B}$

α_β $\qquad\qquad\qquad\qquad\qquad\qquad$ $f(\beta)$

Def: Let X be a topological space and let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X. We say $x \in X$ is a cluster point of the net if for every open set U containing x and for every $\alpha_0 \in A$ there exists $\alpha \in A$ with

$$\alpha \geq \alpha_0 \quad \text{and} \quad x_\alpha \in U.$$



α_0

$\alpha \geq \alpha_0$

Def: We say a net $\langle x_\alpha \rangle_{\alpha \in A}$ is frequently in U if for every $\alpha_0 \in A$ there is $\alpha \geq \alpha_0$ with $x_\alpha \in U$.

$x_n = (-1)^n$ $\qquad$ $U = (\frac{1}{2}, \frac{3}{2})$

$n \geq n_0$ $\qquad$ $x_n \in U$