Prop: A sequentially compact 2^{nd} countable space is compact.

Pf: Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of the sequentially compact 2^{nd} countable space X. Second countable spaces are Lindelöf so we can extract a countable subcover, say, $\{U_i\}_{i=1}^{\infty}$.

Suppose to the contrary that this countable subcover does not admit a finite subcover.

The set U_1 does not cover X so I can pick $x_1 \notin U_1$.

The sets U_1 and U_2 do not cover X so we can pick $x_2 \notin U_1 \cup U_2$.

Continuing inductively we can find $x_j \notin \bigcup_{k=1}^{j} U_k$.

Because X is sequentially compact we can extract a subsequence x_{j_k} converging to some $x \in X$. Since the $\{U_j\}$ covers X we can find J with $x \in U_J$.

Because $x_{j_k} \to x$ there is K so that if $k \geq K$ $x_{j_k} \in U_J$. Observe that if $k \geq J$ $j_k \geq j_J \geq J$ and hence $x_{j_k} \notin U_1 \cup U_2 \cup \cdots \cup U_J$.

In particular if $k \geq K$ and $k \geq J$ then $x_{j_k} \in U_J$ and $x_{j_k} \notin U_J$.

This is a contradiction. $\square$

Corresponding result: sequentially compact + metrizable $\Rightarrow$ compact.

a) See text

b) take real analysis.

Nets: (generalized sequences)

$$\mathbb{N} \to X \qquad x(i) = x_i$$

a sequence is such a map

Def: A directed set is a set A together with a relation $\leq$ satisfying

1) $\alpha \leq \alpha$ for all $\alpha \in A$ (reflexive)
2) If $\alpha \leq \beta$ and $\beta \leq \gamma$ then $\alpha \leq \gamma$ (transitive)
3) If α and $\beta \in A$ there is $\gamma \in A$ with $\gamma \geq \alpha$ and $\gamma \geq \beta$.

Examples

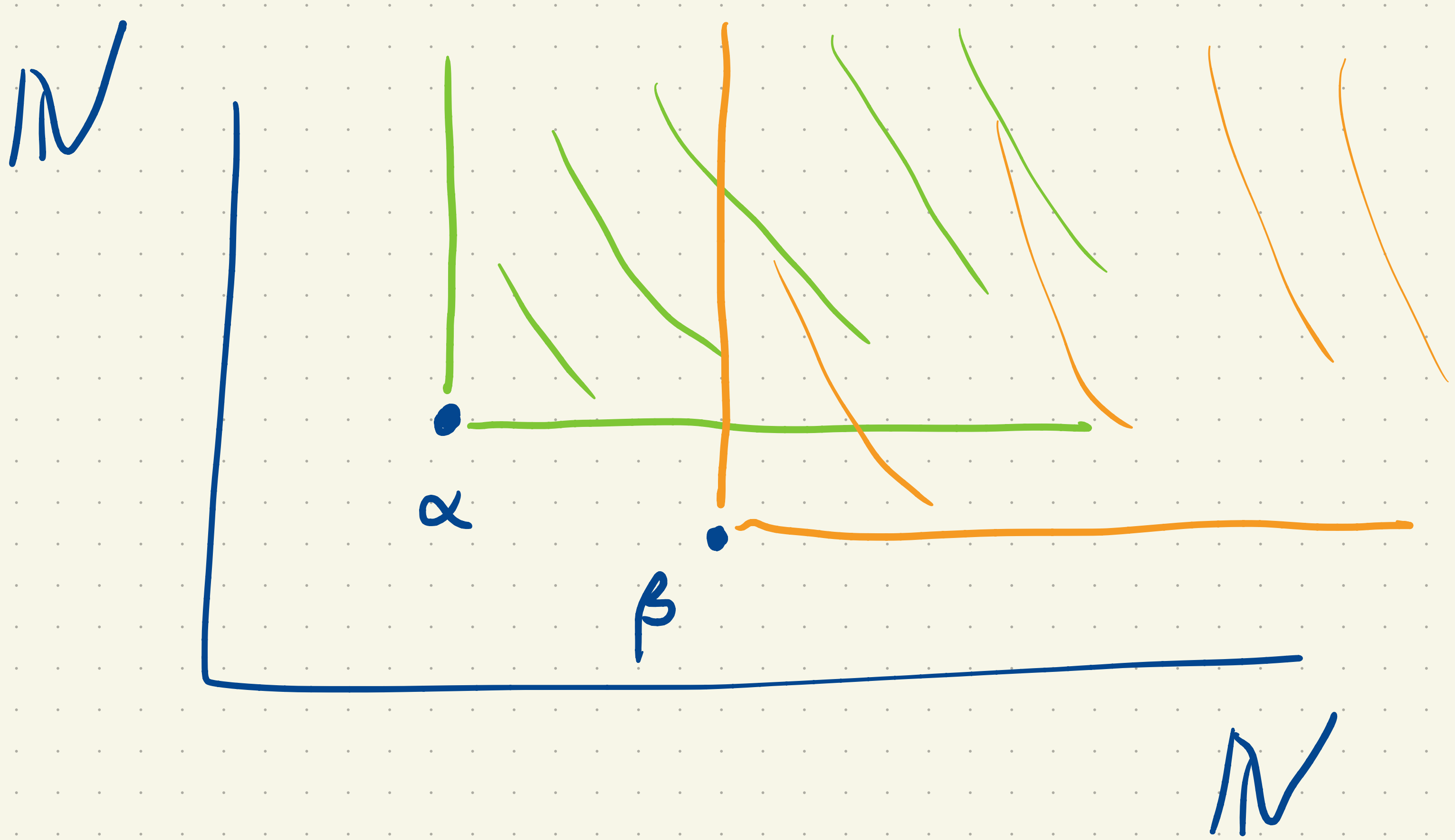
1) $\mathbb{N}, \leq$

$n_1 \quad n_2$

$\max(n_1, n_2)$

2) $\mathbb{N} \times \mathbb{N}$

$(a,b) \leq (c,d)$ if $a \leq c$, $b \leq d$



3) A is a directed set

$A \times A$ is also a directed set

$(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq a_2$
$b_1 \leq b_2$

Given $\alpha = (a_1, b_1)$ and $\beta = (a_2, b_2)$

is there $\gamma \geq \alpha$ $\gamma \geq \beta$. $\gamma = (\gamma_1, \gamma_2)$

Pick γ_1 with $\gamma_1 \geq a_1$ and $\gamma_1 \geq a_2$.

Pick γ_2 with $\gamma_2 \geq b_1$ and $\gamma_2 \geq b_2$

4) X a topological space

$x \in X$

$A = \mathcal{U}(x)$ (the set of all open sets containing x).

U, V $\qquad$ $U \geq V$ if $U \subseteq V$

ordering by reverse inclusion.

$U \geq U$? $\qquad$ $U \subseteq U$?

$U \geq V,\ V \geq W \Rightarrow U \geq W$? $\qquad$ $U \subseteq V,\ V \subseteq W \Rightarrow U \subseteq W$ ✓

U, V $\qquad$ $W = U \cap V$ $\qquad$ $W \in \mathcal{U}(x)$ $\qquad$ $W \subseteq U \Rightarrow W \geq U$

$W \subseteq V \Rightarrow W \geq V$.

Def: Let X be a set. A *net* in X is a function from a directed set A into X.

Are sequences nets? $\mathbb{N} \to X$

Notation: $x(\alpha)$ with $\alpha \in A$

$$x(\alpha) \updownarrow x_\alpha$$

$$\langle x_\alpha \rangle_{\alpha \in A} \qquad \{x_n\}_{n=1}^{\infty} \qquad \{x_n\}_{n \in \mathbb{N}}$$

Many topological properties can be characterized in terms of nets.

E.g. In metric spaces we can characterize the closure of a set $V \subseteq X$ using sequences.

$$x \in \overline{V} \iff \text{there is a sequence in } V \text{ whose limit is } x.$$

Prop: Let X be a topological space and let $V \subseteq X$.
Then $x \in \overline{V}$ if and only if there is a net in V converging to x.

Def: Let A be a directed set and let $\alpha_0 \in A$.
The tail of α_0 in A $T(\alpha_0)$ is $\{\alpha \in A : \alpha \geq \alpha_0\}$.

Exercise: $T(\alpha_0)$ is itself a directed set under the same ordering

$A = \mathbb{N}$ $\quad T(5) = \{n \in \mathbb{N} : n \geq 5\}$

Def: Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X.

A tail of the net is a net of the form

$\langle x_\alpha \rangle_{\alpha \in T(\alpha_0)}$ for some $\alpha_0 \in A$.

Def: Let X be a topological space, $x \in X$, and $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X. We say $x_\alpha \to x$ ($\langle x_\alpha \rangle_{\alpha \in A}$ <u>converges</u> to x)

if for every open set U containing x, U contains all terms of a tail $\langle x_\alpha \rangle_{\alpha \in T(\alpha_0)}$.

Remark: Convergence can be formulated: For all open sets U containing x, there exists α_0 such that if $\alpha \geq \alpha_0$ then $x_\alpha \in U$.

Lemma: Suppose $\langle x_\alpha \rangle_{\alpha \in A}$ is a net in $V \subseteq X$ converging to x.

Then $x \in \overline{V}$.

Pf: We will show x is a contact point of V.

Let U be an open set containing x. Since $x_\alpha \to x$

there exists α_0 so that if $\alpha \geq \alpha_0$, $x_\alpha \in U$.

In particular $x_{\alpha_0} \in U$. Since $x_{\alpha_0} \in V$, x is a contact point.

$\square$

Lemma: Let X be a topological space and $V \subseteq X$.

If $x \in \overline{V}$ then there exists a net in V converging to x.

Pf: Let $A = \mathcal{V}(x)$ ordered by reverse inclusion

For each $U \in A$ there exists some $x_U \in V$, since and $x_U \in U$.

x is in $\overline{V}$ and is hence a contact point of V.

We now have a net $\langle x_U \rangle_{U \in \mathcal{V}(x)}$.

I claim this net converges to x.

Let W be an open set containing x.

Observe that if $U \geq W$ (so $U \subseteq W$)

$x_U \in U \subseteq W$.

So W contains the tail of the net of terms x_U with $U \geq W$.

$\square$