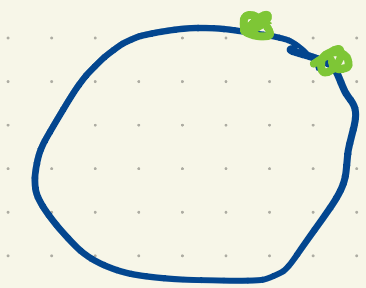
Proof of lemma:

Let $A = \bigcup A_\alpha$. Suppose U and V are disjoint open sets in A with $A = U \cup V$. We need to show that one of U and V is A and the other is empty.

Each A_α is connected in the subspace ~~topology~~ inherited from X and hence also from A. Therefore each A_α is contained in one of U or V. Moreover, if some $A_\alpha \subseteq U$, then since each $A_{\alpha'} \cap A_\alpha \neq \emptyset$ we have $A_{\alpha'} \subseteq U$. Hence, in this case, $\bigcup A_\alpha \subseteq U$. (The case where some $A_\alpha \subseteq V$ is proven identically).

Prop: Suppose $A \subseteq X$ is connected and

$$A \subseteq B \subseteq \overline{A}$$



Then B is connected.

Cor: The closure of a connected set is connected.

Cor: Any closed interval $[a,b]$ is connected.

(each (a,b) is connected)

(a,∞)

$(a,b]$ $\quad$ $[a,\infty)$

Def: A subset $I \subseteq \mathbb{R}$ is an interval if

for all $a, b \in I$ with $a < b$ and all

$c \in \mathbb{R}$ with $a < c < b$ $\quad c \in I$.

(a,b), ϕ, $\mathbb{R}$, $[a,a]$, $[a,b]$ $(a,b]$, $[a,b)$

(a,∞) $\quad (-\infty,b)$ $\quad [a,\infty)$ $\quad (-\infty,b]$

Exercise: show that every interval is one of those.

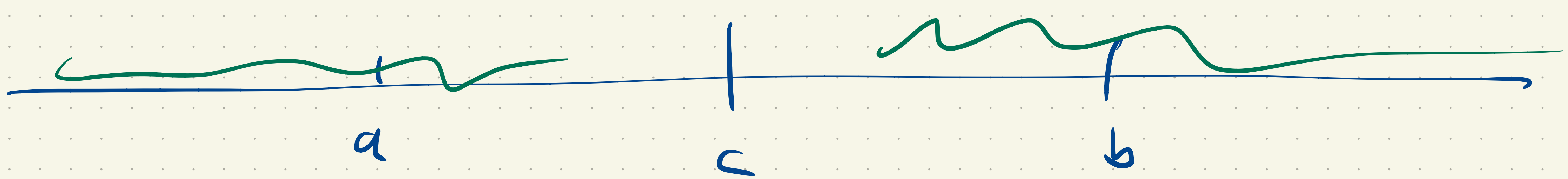
Every interval is connected.

Lemma: Every connected subset of $\mathbb{R}$ is an interval.

(and hence the connected subsets of $\mathbb{R}$ are precisely the intervals.

Pf: Suppose $A \subseteq \mathbb{R}$ is not an interval. Hence there exist

$a, b \in A$ with $a < b$ and c with $a < c < b$ and $c \notin A$,



Let $U = (-\infty, c) \cap A$ and $V = (c, \infty) \cap A$.

Clearly U and V are disjoint and are open in A.

They are nonempty since $a \in U$ and $b \in V$. Since $U \cup V =$

$$= (\mathbb{R} \setminus \{c\}) \cap A$$
$$= A$$

since $c \notin A$, the sets U and V are a separation of A.

(IVT)

Cor: Suppose $f : I \to \mathbb{R}$ is continuous where $I \subseteq \mathbb{R}$ is an interval. Then $f(I)$ is an interval.

Pf: Observe that $f(I)$ is connected since intervals are connected and since the continuous image of a connected set is connected. Hence $f(I)$ is an interval.

$cl_X(A)$ in X

Proof of Prop ($A \subseteq B \subseteq \overline{A}$, A connected $\Rightarrow$ B is connected)

Suppose U and V are disjoint open subsets of B that cover B (i.e. $B = U \cup V$). Job: Show one is empty (and the other is B)

Since $A \subseteq B$ is connected in B it is contained in one of U or V. WLOG we can assume $A \subseteq U$. Observe that U is closed in B (its complement is V which is open) Therefore $cl_B(A) \subseteq U$.

$A \subseteq B \subseteq X$

But $cl_B(A) = cl_X(A) \cap B = B$.

So $U \supseteq cl_B(A) = B$. $\square$

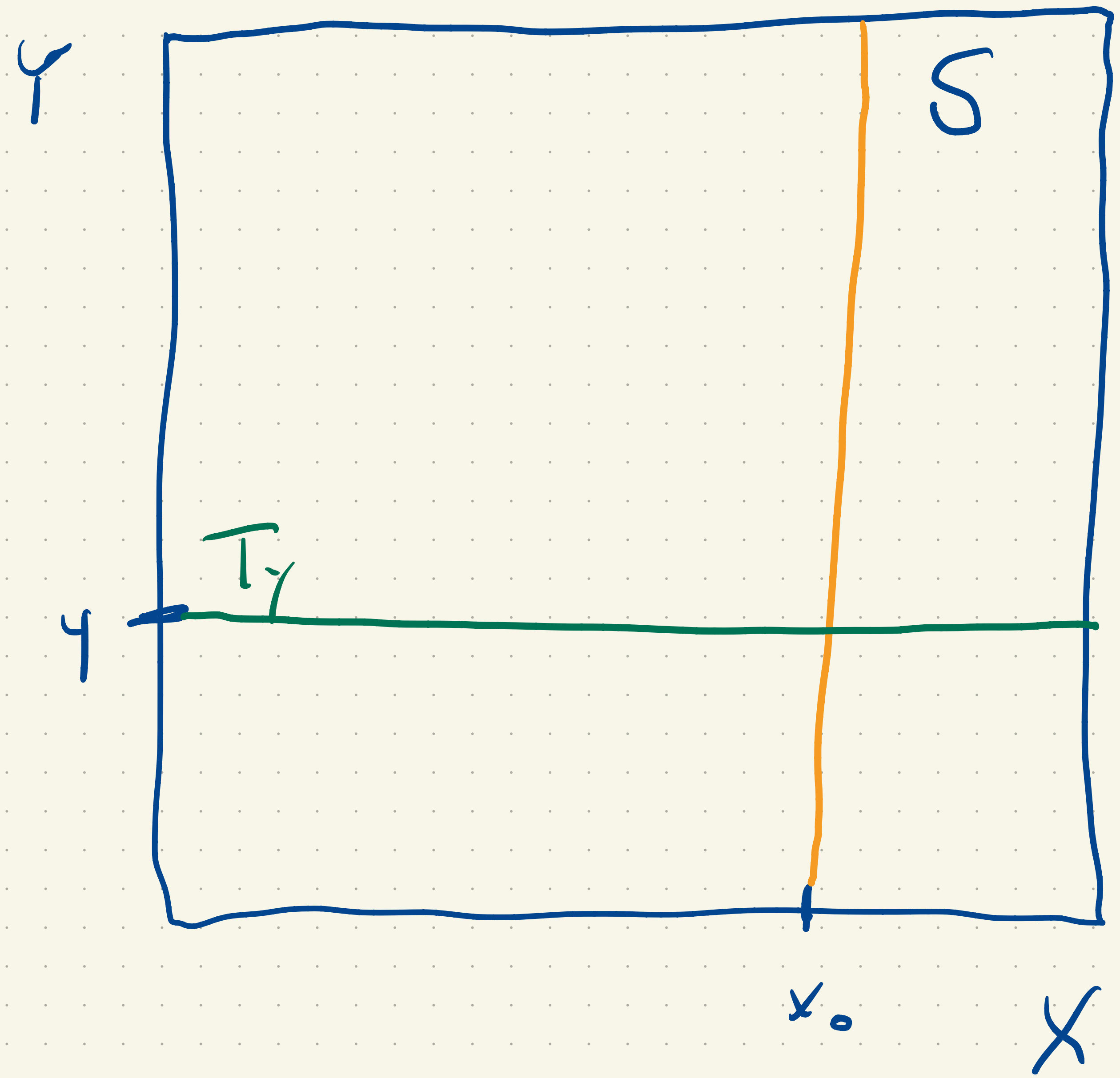
Prop: A quotient of a connected space is connected.

q.m. $\Rightarrow$ cts

$$X \xrightarrow{\pi} Y \qquad \pi(X) = Y$$

connected

Products? I claim that if X, Y are connected then $X \times Y$ is also connected.



$X \times Y$

$X \neq \emptyset$

$Y \neq \emptyset$

Pick $x_0 \in X$

$S = \pi_X^{-1}(\{x_0\})$

Claim S is connected

$i_{x_0} : Y \to S$

$i_{x_0}(y) = (x_0, y)$

$i_{x_0}(Y) = S$

For each $y \in Y$, let

$T_y = \pi_Y^{-1}(\{y\})$

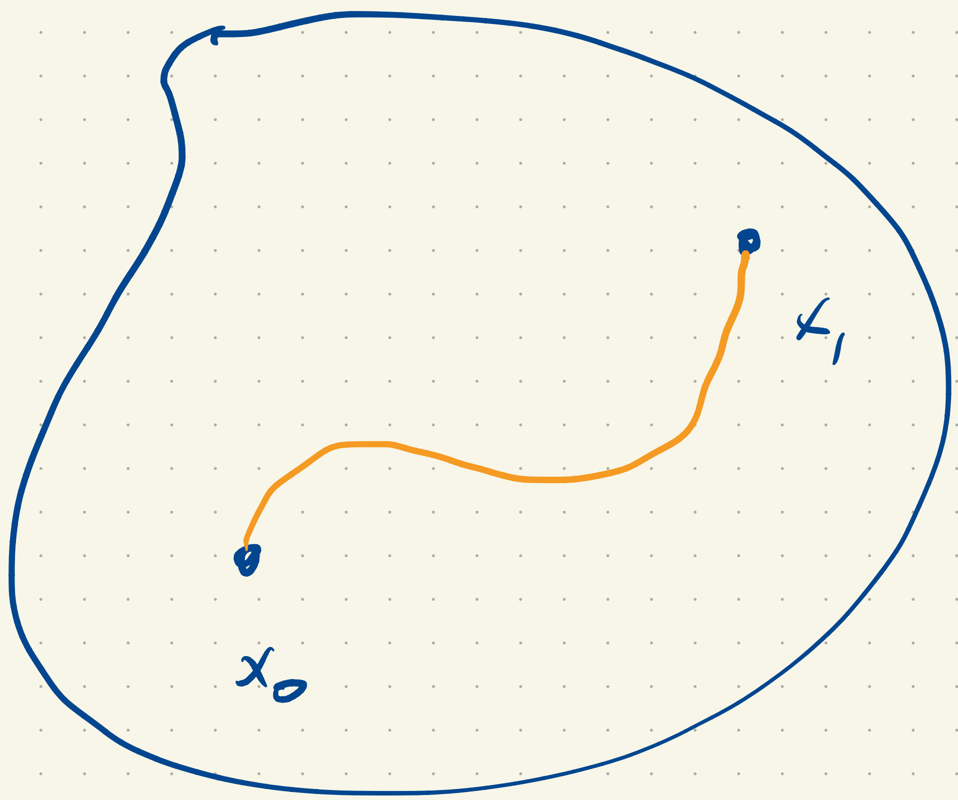
Observe each T_y is connected by a similar argument using the connectivity of X.

Let $R_y = T_y \cup S$. Then R_y is connected

since each of S and T_y are and since $S \cap T_y = \{(x_0, y)\}$

Observe $\bigcup R_y = X \times Y$ and $\bigcap R_y = S \neq \emptyset$ since $Y \neq \emptyset$.

So $X \times Y$ is a union of connected sets with a point in common.


x_1
x_0