$\pi : X \to Y$

Lemma: A quotient of a Lindelöf space is Lindelöf.

Consequence: If $\pi : X \to Y$ is a quotient map and X is 2^{nd} countable and Y is locally Euclidean then Y is 2^{nd} countable.

2^{nd} countable $\Rightarrow$ Lindelöf

$\Rightarrow$ Y is Lindelöf + locally Euclidean

$\Rightarrow$ 2^{nd} countable.

Pf of Lemma:

Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of Y.

Consider the sets $\{\pi^{-1}(U_\alpha)\}_{\alpha \in I}$. Observe

$$X = \pi^{-1}(Y) = \pi^{-1}\Big(\bigcup_{\alpha \in I} U_\alpha\Big) = \bigcup_{\alpha \in I} \pi^{-1}(U_\alpha)$$

and hence we have an open cover of X.

Since X is Lindelöf we can reduce to a countable subcover $\{\pi^{-1}(U_{\alpha_k})\}_{k=1}^{\infty}$

Then $Y = \pi(X) = \pi\left(\bigcup_k \pi^{-1}(U_{\alpha_k}) \right)$

surjective

$$= \pi\left(\pi^{-1}\left(\bigcup_k U_{\alpha_k}\right)\right)$$

surjectivity

$$= \bigcup_k U_{\alpha_k}$$

So $\{U_{\alpha_k}\}$ is a countable subcover.

Connectness:

Def: Let X be a top space. A <u>separation</u> of X is a pair of disjoint nonempty open sets U, V such that $U \cup V = X$. A space is <u>disconnected</u> if it admits a separation, otherwise it is <u>connected</u>.

E.g. $\mathbb{Z} \subseteq \mathbb{R}$

$$U = \{ z \in \mathbb{Z} : z > 1 \}$$
$$V = \{ z \in \mathbb{Z} : z \leq 1 \}$$

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$\emptyset$ connected!

$\mathbb{Q}$

$$U = \mathbb{Q} \cap (-\infty, \pi)$$
$$V = \mathbb{Q} \cap (\pi, \infty)$$

Prop $\mathbb{R}$ is connected

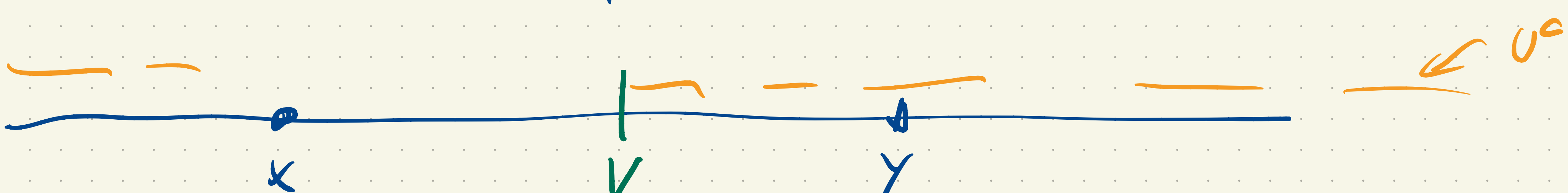
Pf: Suppose $U \subseteq \mathbb{R}$ is open, $U \neq \emptyset$, $U \neq \mathbb{R}$.

Job: show U^c is not open.

Pick $x \in U$ and pick $y \in U^c$. We will assume $x < y$. The case $y < x$ is proved similarly.

Let $W = \{ w \in U^c : x < w \}$.

Observe: $W \neq \emptyset$ as $y \in W$ and W is bounded below by x. Hence W admits an infimum v.



From elementary analysis each set $[v, v+\varepsilon)$ intersects W.

In particular $(v-\varepsilon, v+\varepsilon)$ intersects $W^{v+\varepsilon}$ for each $\varepsilon > 0$

and therefore $v \notin \text{Int}(U)$. But U is open, so $v \in U^c$.

Since $v = \inf W$, the entire interval $[x, v)$ lies in U.

Hence any interval $(v-\varepsilon, v+\varepsilon)$ intersects U and hence

$v \notin \text{Int}(U^c)$. Since $v \in U^c$, U^c is not open

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If X is homeomorphic to Y and X is connected

so is Y. $\Rightarrow$ connectedness is a topological property

Cor: open intervals in $\mathbb{R}$ are connected.

Prop: If X is connected and $f: X \to Y$ is continuous and surjective then Y is connected.

"The continuous image of a connected set is connected"

Pf: Suppose $f: X \to Y$ is continuous and Y is disconnected.

Job: Show X is disconnected.

Let U, V be a separation of Y.

Consider $f^{-1}(U)$, $f^{-1}(V)$.

These are: • open continuity

- nonempty surjectivity

- a cover of X

$$X = f^{-1}(Y) = f^{-1}(U \cup V)$$
$$= f^{-1}(U) \cup f^{-1}(V).$$

So they form a separation of X. $\square$

$[-1, 1]$ is connected because it is $\sin(\mathbb{R})$

Remark: A space X is connected iff the only subsets of X that are both open and closed are X and $\emptyset$.

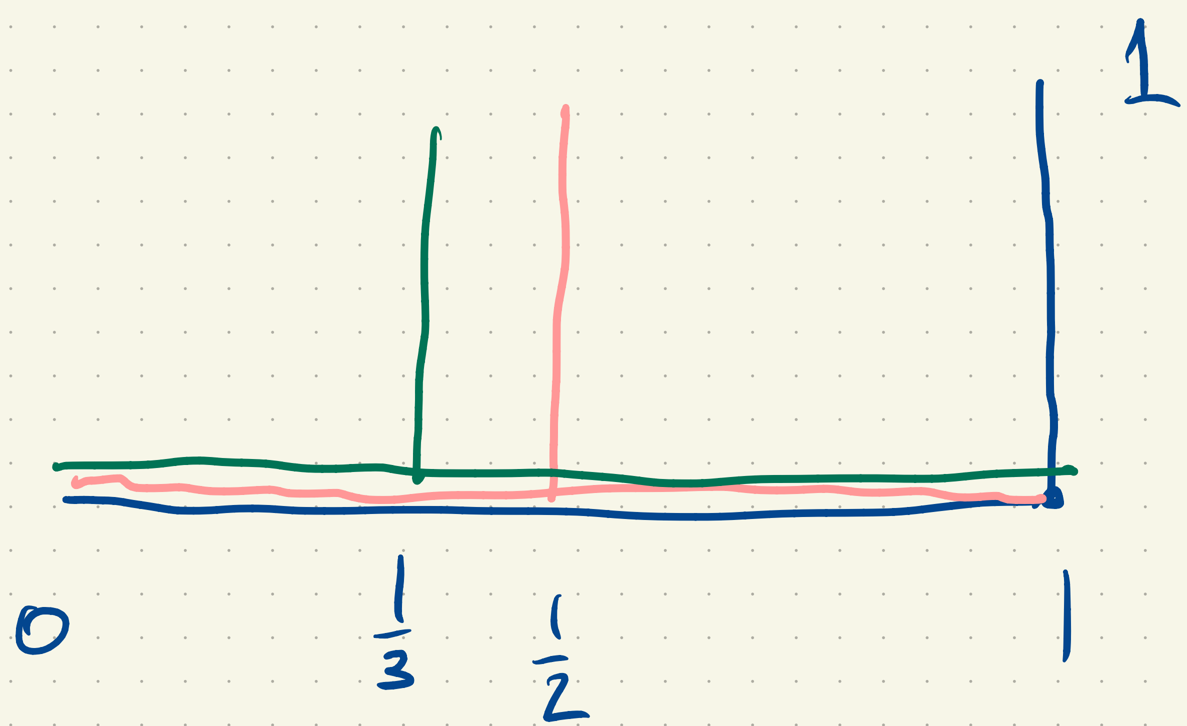
Prop: If $A \subseteq X$ is connected and if U and V are disjoint open sets in X such that $A \subseteq U \cup V$ then either $A \subseteq U$ or $A \subseteq V$.

Pf: If $A \cap V$ and $A \cap U$ are both nonempty then they form a separation of A.

Prop: Suppose $\{A_\alpha\}_{\alpha \in I}$ is a collection of connected sets in X.

If $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$ then $\bigcup A_\alpha$ is connected.

(A union of connected sets with a point in common is connected)



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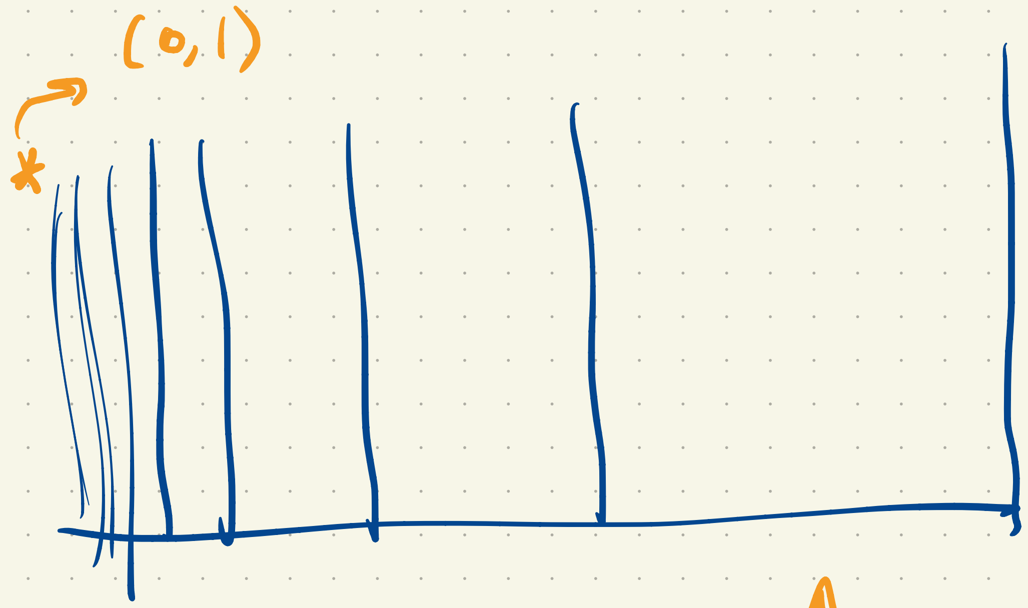

A, connected.

comb space