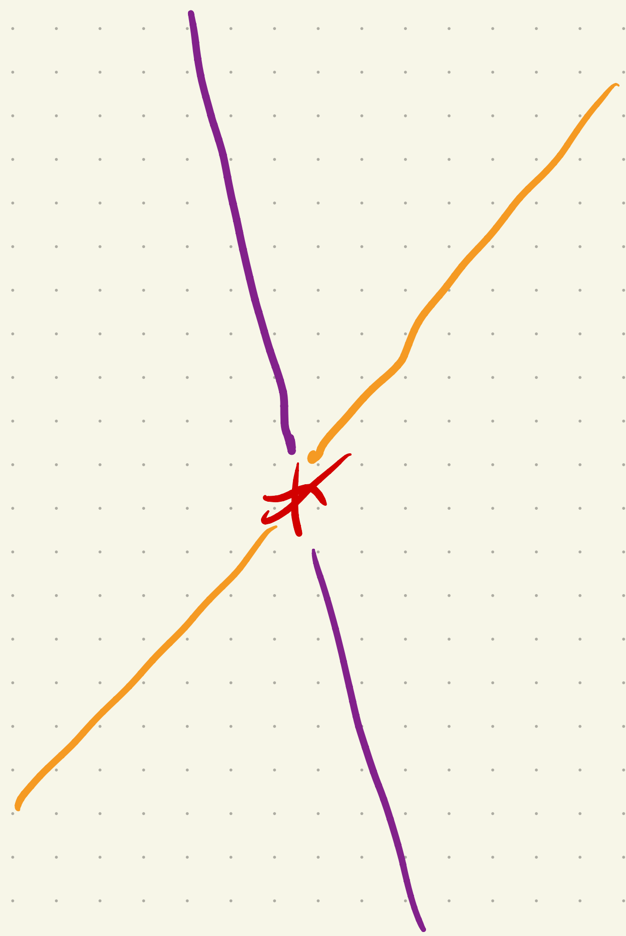
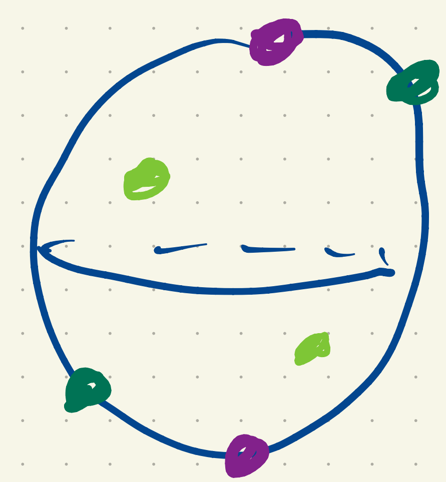
$\mathbb{R}^{n+1,*}$ $\quad x \sim y$ if $x = \lambda y$ $\quad \lambda \neq 0$



$$\mathbb{RP}^n = \mathbb{R}^{n+1,*}/\sim$$

projective space



Claim: $\mathbb{RP}^n \simeq S^n/\sim$ where $x \sim -x$

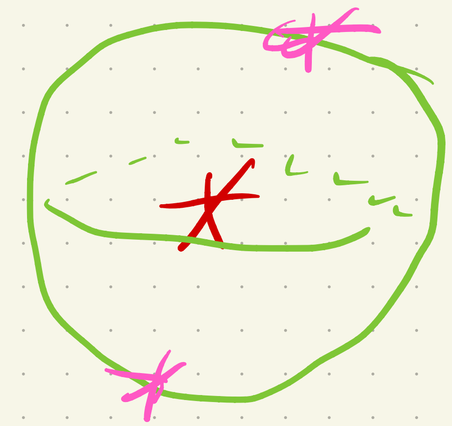
$$\begin{tikzcd}
S^n \arrow[r, hook, "\iota"] \arrow[d, "\pi_2"'] \arrow[dr, "\tilde f"] & \mathbb{R}^{n+1,*} \arrow[d, "\pi_1"] \\
S^n/\sim \arrow[r, "f"'] & \mathbb{RP}^n
\end{tikzcd}$$

$$\tilde f(x) \overset{?}{=} \tilde f(-x)$$

$\tilde f$ is const on fibers!

$$\boxed{\begin{aligned}\tilde f(-x) &= \pi_1(\iota(-x)) \\ &= \pi_1(-x) = \pi_1(x) = \pi_1(\iota(x)) \\ &= \tilde f(x)\end{aligned}}$$

$$\begin{tikzcd} \mathbb{R}^{n+1,*} \arrow[r, "n"] \arrow[d, "\pi_1"'] & S^n \arrow[d, "\pi_2"] \\ \mathbb{RP}^n \arrow[r, "g"'] & S^n/\sim \end{tikzcd}$$



$$n(x) = \frac{x}{\|x\|}$$

Is $\pi_2 \circ n$ constant on the fibers of π_1?

$$\pi_2(n(\lambda x)) = \pi_2(n(x)) \quad \text{for any } \lambda \neq 0$$
$$\text{any } x \in \mathbb{R}^{n+1,*}$$

$$\pi_2(n(\lambda x)) = \pi_2\left(\frac{\lambda x}{\|\lambda x\|}\right) = \pi_2\left(\frac{\lambda x}{|\lambda|\|x\|}\right) = \pi_2\left(\pm \frac{x}{\|x\|}\right))$$

$$= \pi_2\left(\frac{x}{\|x\|}\right)$$

$$= \pi_2(n(x))$$

$$\begin{aligned}
f(g(\pi_1(x))) &= f(\pi_2(n(x))) \\
&= f\left(\pi_2\left(\frac{x}{\|x\|}\right)\right) \\
&= \pi_1\left(i\left(\frac{x}{\|x\|}\right)\right) \\
&= \pi_1\left(\frac{x}{\|x\|}\right) \\
&= \pi_1(x)
\end{aligned}$$

$$g(f(\pi_2(p)))$$

$$\shortparallel$$

$$g(\pi_1(i(p)))$$

$$\shortparallel$$

$$g(\pi_1(p))$$

$$\shortparallel$$

$$\pi_2(n(p))$$

$$\shortparallel$$

$$\pi_2(p)$$

$n(p) = p$

$$\begin{tikzcd}
S^n \arrow[r, hook, "i"] \arrow[d, "\pi_2"'] \arrow[dr, "\tilde{f}"] & \mathbb{R}^{n+1,*} \arrow[d, "\pi_1"] \\
S^n/\sim \arrow[r, "f"'] & \mathbb{R}P^n
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R}^{n+1,*} \arrow[r, "n"] \arrow[d, "\pi_1"'] \arrow[dr, "\tilde{g}"] & S^n \arrow[d, "\pi_2"] \\
\mathbb{R}P^n \arrow[r, "g"'] & S^n/\sim
\end{tikzcd}$$