Exercise: $\pi: X \to Y$, a surjection, is a quotient map iff it is continuous and takes saturated closed sets to closed sets.

(continuous surjections that are either open or closed maps are quotient maps)

Eg: $[0,1] \xrightarrow{\mathcal{E}} S^1 \subseteq \mathbb{C}$

$$\mathcal{E}(t) = e^{2\pi i t}$$

I claim $\mathcal{E}$ is a quotient map.

It's evidently continuous and surjective

I claim it is a closed map.

Suppose $V \subseteq [0,1]$ is closed.

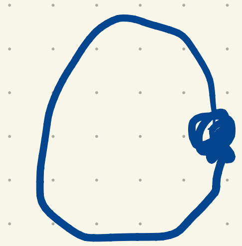
To show $\mathcal{E}(V)$ is closed we need only show that it contains its sequential limit points

Consider a sequential limit point $p$ of $\mathcal{E}(V)$. Then there exists a sequence $p_k$ in $\mathcal{E}(V)$ converging to some $p$.

Job: $p \in \mathcal{E}(V)$.

For each $k$ we can pick $q_k \in V$ with $\mathcal{E}(q_k) = p_k$.

Now $\{q_k\}$ is a sequence in $[0,1]$.



By the B-W theorem there is a subsequence

$q_{k_j} \to q \in [0,1]$ for some $q$. Better than that, because

each $q_{k_j} \in V$ and because $V$ is closed, $q \in V$.

Then $p_{k_j} = \mathcal{E}(q_{k_j}) \to \mathcal{E}(q) \in \mathcal{E}(V)$.

But $p_{k_j} \to p$ as well. So $p \in E(V)$.

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**Thm (CPQT)**

Suppose $\pi : X \to Y$ is a quotient map, $Z$ is a space, and $f : Y \to Z$ is a function. Then

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

$f$ is continuous if and only if $\tilde{f} := f \circ \pi$ is

Thm: (Descending to the quotient)

Suppose $\pi: X \to Y$ is a quotient map and $\tilde{f}: X \to Z$ is constant on the fibers of $\pi$. Then there exists a unique $f: Y \to Z$ such that $\tilde{f} = f \circ \pi$.

Moreover, $f$ is continuous iff $\tilde{f}$ is.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Thm: Uniqueness of Quotients.

Suppose $\pi_i : X \to Y_i$ are quotient maps that make the same identifications $\left( \pi_1(a) = \pi_1(b) \iff \pi_2(a) = \pi_2(b) \right)$.

Then $Y_1$ and $Y_2$ are homeomorphic by the map taking any $\pi_1(x)$ to $\pi_2(x)$.

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y_1 & & Y_2 \end{tikzcd}$$

Pf:

$$\begin{tikzcd} X \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y_1 \arrow[r, dashed, "f_{12}"'] & Y_2 \end{tikzcd}$$

Because $\pi_2$ is constant on the fibers of $\pi_1$, $\pi_2$ descends to a continuous map $f_{12} : Y_1 \to Y_2$

Conversely there is a continuous $f_{21}: Y_2 \to Y_1$ with

$$\begin{tikzcd} X \arrow[d, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ Y_2 \arrow[r, dashed, "f_{21}"'] & Y_1 \end{tikzcd}$$

Observe: $f_{21}(f_{12}(\pi_1(x))) = f_{21}(\pi_2(x)) = \pi_1(x)$.

Since $\pi_1$ is surjective, $f_{21}(f_{12}(y)) = y$ for all $y \in Y_1$.

The argument that $f_{12}(f_{21}(z)) = z$ for all $z \in Y_2$ is similar.

$\square$

$\underbrace{[0,1]}_{I} / \sim \quad \text{or} \quad S^1$

$$\begin{tikzcd}
 & I \arrow[dl, "\pi"'] \arrow[dr, "\varepsilon"] & \\
I/\sim & & S^1
\end{tikzcd}$$

$\pi$ and $\varepsilon$ are q.m. that make the same identifications

So $I/\sim \; \sim \; S^1$.

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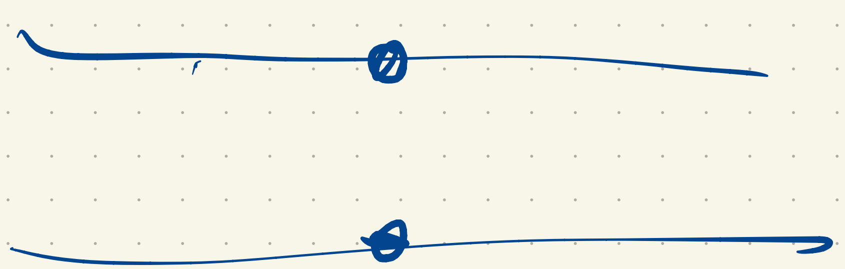
Quotient maps are awful: Quotients of Hausdorff spaces need not be Hausdorff

Quotients of locally euclidean spaces need not be loc. euc.



Quotients of manifolds need not be manifolds

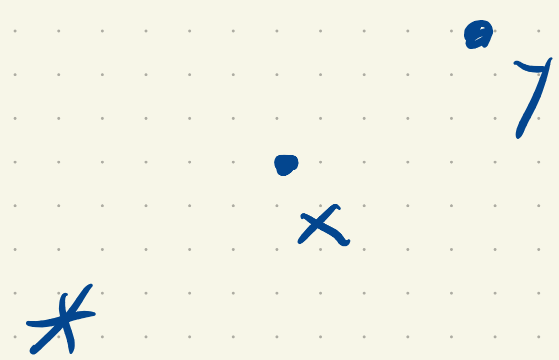
Exercise: A quotient of a Lindelöff space is Lindelöff.

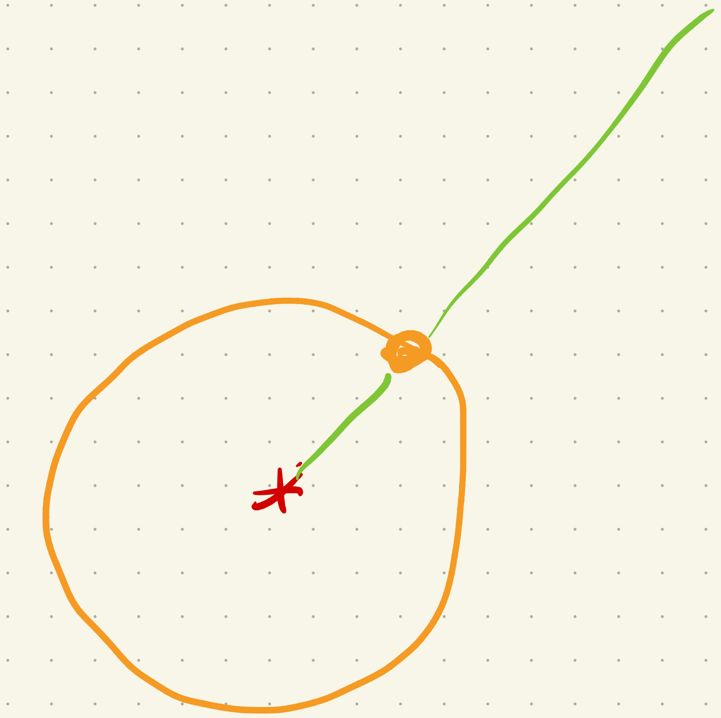
Exercise: If $\pi : X \to Y$ is a quotient map and $X$ is $2^{nd}$ countable and $Y$ is locally Euclidean then $Y$ is $2^{nd}$ countable.

Hint

$$\mathbb{R}^{n+1,*} = \mathbb{R}^{n+1} \setminus \{0\}$$

$x \sim y$ if $\exists\ \lambda > 0$ with $x = \lambda y$.



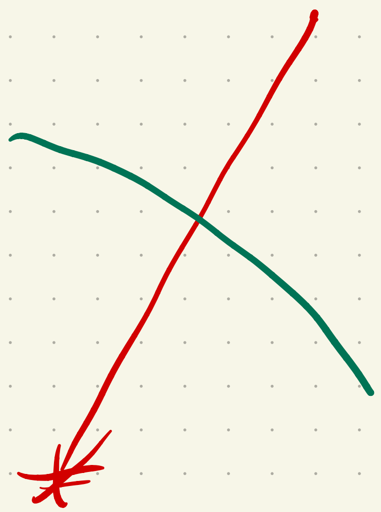



Claim $\underbrace{\mathbb{R}^{n+1,*}/\sim}_{\textstyle\longrightarrow Y}$ is homeomorphic to $S^n$?

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[dr, "g"'] & \mathbb{R}^{n+1,*} \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

$g$ is cts as it is a composition of cts functions

$$\begin{tikzcd} \mathbb{R}^{n+1,*} \arrow[d, "\pi"'] \arrow[dr, "\tilde f"] & \\ Y \arrow[r, "f"'] & S^n \end{tikzcd}$$



$$\tilde f(x) = \frac{x}{\|x\|}$$

Is $\tilde f$ constant on the fibers of $\pi$?

$\lambda > 0$

$$\tilde f(\lambda y) = \frac{\lambda y}{\|\lambda y\|} = \frac{\lambda y}{|\lambda|\,\|y\|} = \frac{y}{\|y\|} = \tilde f(y)$$

$\tilde f(\lambda y) = \tilde f(y) \Rightarrow \tilde f$ is const on fibers.

So $\tilde f$ descends to a continuous map $f : Y \to S^n$.

$$f(g(x)) = f(\pi(i(x))) = f(\pi(x)) = \tilde f(x) = \frac{x}{\|x\|} = x$$

some $x \in S^1$.

$$g(f(\pi(x))) = g(\tilde{f}(x))$$

$$= g\left(\frac{x}{\|x\|}\right)$$

$$= \pi\left(i\left(\frac{x}{\|x\|}\right)\right)$$

$$= \pi\left(\frac{x}{\|x\|}\right)$$

$$= \pi(x) \quad \text{since } \frac{1}{\|x\|} > 0.$$

Hence $g = f^{-1}$.