$$\begin{tikzcd} X \arrow[d, "\pi"'] & p \\ X/\sim & \end{tikzcd}$$

Def: Let $q \in X/\sim$. The fiber over q is the set $\pi^{-1}(\{q\}) = [p]$

(It's an equivalence class) $\pi(p) = q$

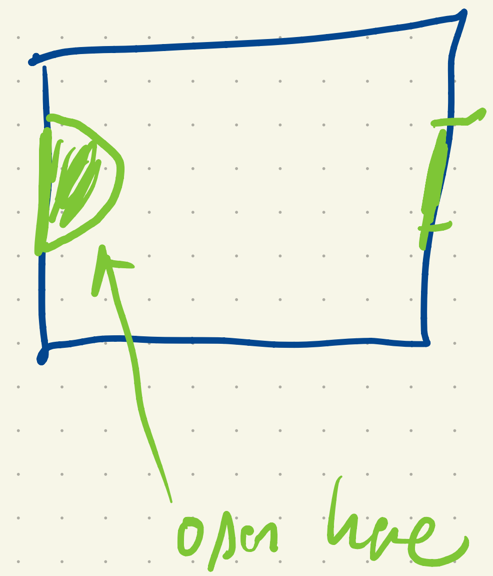
How do we visualize sets in $X/\sim$?

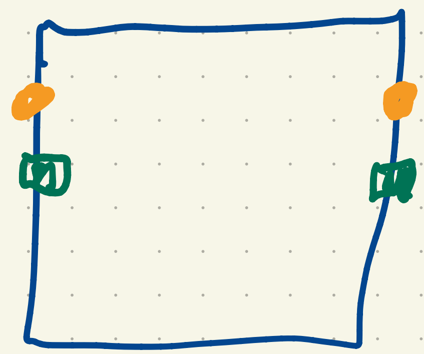
Unions of fibers in X.

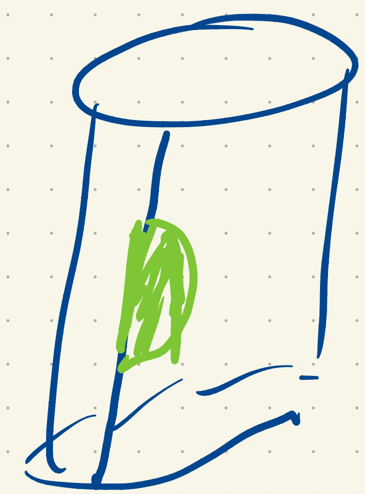
Def: A set $V \subseteq X$ is <u>saturated</u> with respect to π if there exists $A \subseteq X/\sim$ where $V = \pi^{-1}(A)$

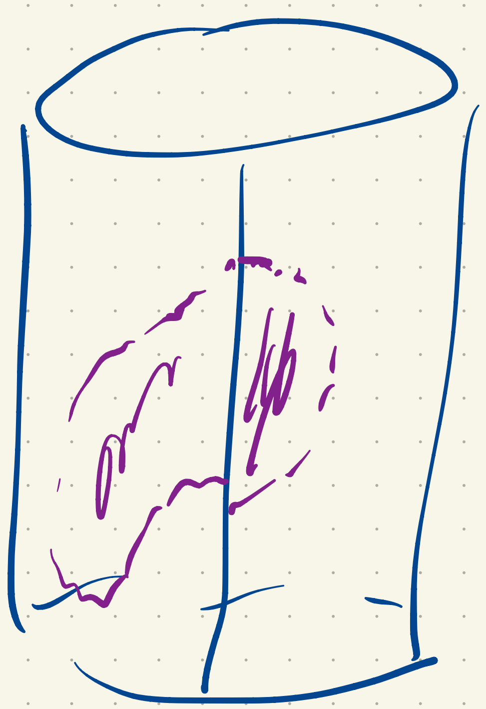
(exactly when V is a union of fibers) $\bigcup_{q\in A} \pi^{-1}(\{q\})$

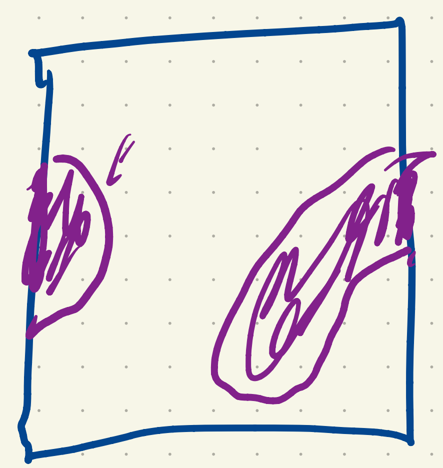
$I \times I \quad (0,y) \sim (1,y)$











$f \circ \pi$

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ X/\sim \arrow[r, "f"'] & Y \end{tikzcd}$$

$\tilde{f} = f \circ \pi$

CPQT: f is continuous iff $\tilde{f}$ is.

quotient

If f is continuous then so is $\tilde{f}$, clearly (composition of cts maps!)

Suppose $\tilde{f}$ is continuous. Consider an open set $U \subseteq Y$.

Note $f^{-1}(U)$ is open in $X/\sim$ iff $\pi^{-1}(f^{-1}(U))$

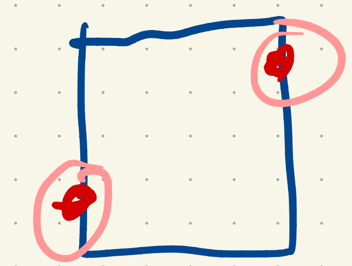
is open in X. But $\pi^{-1}(f^{-1}(U)) = (f\circ\pi)^{-1}(U)$

$$= \tilde{f}^{-1}(U)$$

which is open in X by continuity of $\tilde{f}$.

Hence $f^{-1}(U)$ is open in $X/\sim$.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ X/\sim \arrow[r, "f"'] & Y \end{tikzcd}$$



we require

if $\pi(p_1) = \pi(p_2)$

then $\tilde{f}(p_1) = \tilde{f}(p_2)$

$$f([p]) := \tilde{f}(p)$$

" $\tilde{f}$ is constant on the fibres of π "

$$\pi^{-1}(\{q\})$$

With this restriction, $\tilde{f}$ defines a function f on $X/\sim$ by the above rule.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ X/\sim \arrow[r, "f"'] & Y \end{tikzcd}$$

If $\tilde{f}$ is constant on the fibres of π there exists a unique $f : X/\sim \to Y$ such that the diagram commutes. Moreover, f is continuous if and only if $\tilde{f}$ is.

" $\tilde{f}$ descends to the quotient"

Generalization

$$\pi : X \longrightarrow Y$$

top. space $\to X$, set $\to Y$

π surjection

$$\begin{array}{c} X \\ \pi \downarrow \\ X/\sim \end{array}$$

Def: The quotient topology on Y is defined by

$$\{ U \subseteq Y : \ \pi^{-1}(U) \text{ is open in } X \}$$

$$\begin{tikzcd} & X \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \\ X/\sim & & Y \end{tikzcd}$$

fibers, saturated sets all mean the same thing in this context.

$$p_1 \sim p_2 \iff \pi(p_1) = \pi(p_2)$$

$$X \xrightarrow{\pi} Y \qquad \pi \text{ surjection}$$

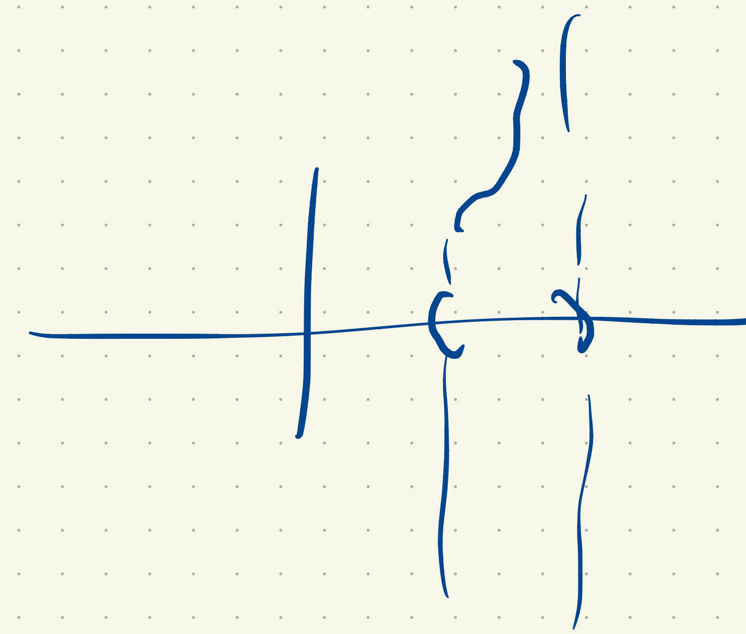
space   space

$$\begin{array}{c} X \\ \downarrow \pi \\ Y \end{array}$$

We say π is a <u>quotient</u> map if the topology on Y is the same as the quotient topology induced by π.

$\pi^{-1}(A)$

$A \subseteq Y$



$$\begin{array}{c} \mathbb{R}^2 \\ \pi_1 \downarrow \\ \mathbb{R} \end{array} \qquad \pi_1(x,y) = x$$

It turns out that π_1, π_2 are quotient maps

This is an open map

Prop: $\pi : X \to Y$, a surjection is a quotient map iff it is continuous and takes saturated open sets to open sets.

Pf: Suppose π is a quotient map. Then it's continuous. Consider a saturated open set $W \subseteq X$. Then there is a set $A \subseteq Y$ such that $W = \pi^{-1}(A)$. Moreover, because π is surjective $\pi(\pi^{-1}(A)) = A$. Since $\pi^{-1}(A)$ is open in X, A is open in Y.

Conversely: suppose π is continuous and takes saturated open sets to open sets.

We want to show π is a quotient map which means showing that a set $A \subseteq Y$ is open if and only if $\pi^{-1}(A)$ is open in X.

Suppose $A \subseteq Y$ is open. Then $\pi^{-1}(A)$ is open in X since π is continuous.

Suppose $A \subseteq Y$ and $\pi^{-1}(A)$ is open in X.

Then $\pi^{-1}(A)$ is a saturated open set and $\pi(\pi^{-1}(A))$ is open in Y. But $\pi(\pi^{-1}(A)) = A$ again using surjectivity.

So A is is open in Y.