Product of Manifolds is a manifold

$$A = M^{d_1} \times N^{d_2}$$

claim: A is locally Euclidean of dimension $d_1 + d_2$

Pick $(p, q) \in A$.

We can find open sets U and V contains p and q respectively

with homeomorphisms $\phi : U \to \mathbb{R}^{d_1}$ $\psi : V \to \mathbb{R}^{d_2}$.

Define $\Phi : U \times V \to \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ by $\quad (\Phi = \phi \times \psi)$

$$\Phi(x, y) = (\phi(x), \psi(y))$$

Claim Ψ is continuous.

xypic

$$\begin{tikzcd}
U \times V \arrow[r, "\Psi"] \arrow[d, "\pi_U"'] & \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \arrow[d, "\pi_1"] \\
U \arrow[r, "\phi"] & \mathbb{R}^{d_1}
\end{tikzcd}$$

Since $\phi \circ \pi_U$ is continuous so is $\pi_1 \circ \Psi$.

Similarly $\pi_2 \circ \Psi$ is cts.

So by CPPT, Ψ is cts.

Is Ψ^{-1} continuous?

$$\Psi^{-1} : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to U \times V$$

$$\Psi^{-1} = \phi^{-1} \times \psi^{-1}$$

By the above $U \times V$ with the product topology is homeomorphic to $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

Exercise $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \sim \mathbb{R}^{d_1+d_2}$

Exercise $U \times V$ with the product topology is homeomorphic to $U \times V$ with the subspace topology

"A product of subspaces is a subspace of products"

$$(U \times V)_p \xrightarrow{Id_{p,s}} (U \times V)_s$$

$$\to M^{d_1} \times M^{d_2}$$

$\{X_\alpha\}_{\alpha\in A}$

$$\prod_{\alpha\in A} X_\alpha = \{ f : A \to \bigcup_{\alpha\in A} X_\alpha : f(\alpha)\in X_\alpha \}$$

If all X_α's are same X then

X^2 $\quad A = \{0,1\}$

(a,b)

$X^n \longleftrightarrow A = \underbrace{\{0,1,2,\dots,n-1\}}_{n}$

$X^\omega \longleftrightarrow A = \mathbb{N}$

$X^Y \longleftrightarrow A = Y$

X^Y is the set of all maps from Y to X

What would be topologies to put on $\prod_{\alpha \in A} X_\alpha$?

$U_\alpha \subseteq X_\alpha$

open

$\prod_{\alpha \in A} U_\alpha$ could take these to be a basis.

Resulting topology τ_b is the "box topology"

For finitely many factors the product topology is the ~~weakest~~ coarsest topology such that the projections are continuous If we follow that strategy here we want the coarsest topology such that

$\pi_\alpha^{-1}(U)$ is open in the product

for all $\alpha \in A$ and all $\underset{\text{open}}{\wedge}$ $U \subseteq X_\alpha$.

$$\mathcal{S} = \{ \pi_\alpha^{-1}(U) : \alpha \in \mathcal{A},\ U \subseteq X_\alpha \text{ is open} \}$$

subbasis

Basis by taking finite intersections

A basic open set has the form $\prod_{\alpha \in A} U_\alpha$ where

each U_α is open in X_α <u>and</u> all but finitely many

U_α are X_α.

This is the product topology τ_p.

$$\tau_p \subseteq \tau_b$$

It is strict in general

e.g. $\prod_{n\in\mathbb{N}} \left(\frac{1}{n}, \frac{1}{n}\right)$ is not open in $\mathbb{R}^\omega$ with the product topology.

(Exercise).

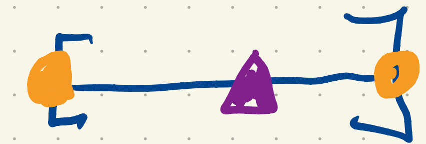
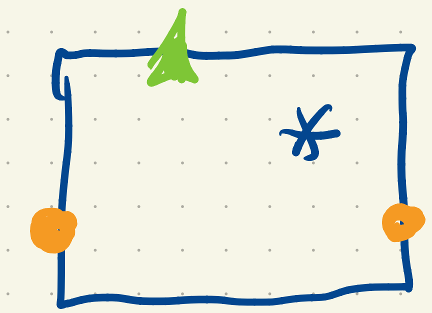
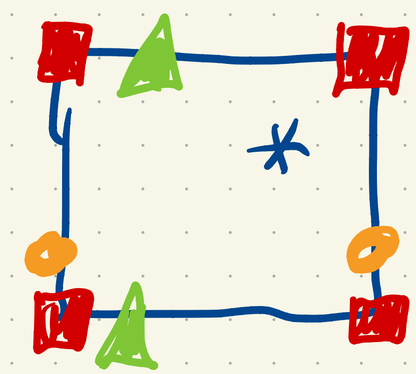
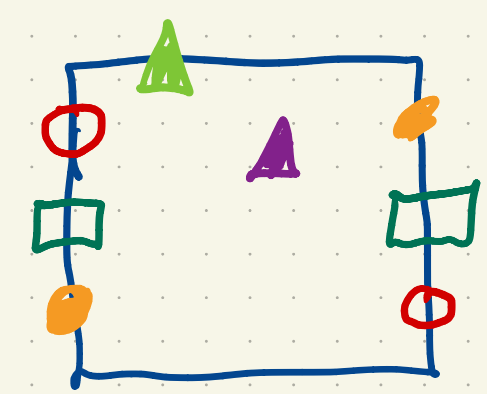
Hint: If U is open in the product topology then $\pi_\alpha(U) = X_\alpha$ for all but finitely many α.

By default: a product gets the product topology.

Check: almost everything we proved about the product topology goes over to the case of arbitrary factors

In particular, it satisfies CPPT (and the CPPT is characteristic)

Topological spaces constructed by gluing.

	X	Equivalence relation		
1)	$I = [0,1]$	$0 \sim 1$		circle
2)	$I \times I$	$(0,y) \sim (1,y)$		cylinder
3)	$I \times I$	$(0,y) \sim (1,y)$ $(x,0) \sim (x,1)$		$S^1 \times S^1$ torus
4)	$I \times I$	$(0,y) \sim (1,1-y)$		Möbius strip

Goal: Find a topology on $X/\sim$ that matches ~~our~~ intuition above.

$A \subseteq X$ wanted $i_A : A \to X$ to be cts.

$A \times B$ wanted $\pi_A : A \times B \to A$, $\pi_B : A \times B \to B$ to be cts.

$$X \xrightarrow{\pi} X/\sim$$

$$x \longmapsto [x]$$

We'd like π to be continuous.

We'll select the richest topology such that π is cts.

$$\tau := \{ U \subseteq X/\sim \; : \; \pi^{-1}(U) \text{ is open in } X \}$$

$$\pi^{-1}\Big(\bigcup_{\alpha \in I} U_\alpha\Big) = \underbrace{\bigcup_{\alpha \in I} \underbrace{\pi^{-1}(U_\alpha)}_{\text{open in } X}}_{\text{open in } X}$$

$$\pi^{-1}\Big(\bigcap_{i=1}^{n} U_i\Big) = \underbrace{\bigcap_{i=1}^{n} \pi^{-1}(U_i)}_{\text{open in } X}$$

τ is a topology

Is $\pi : X \to X/\sim$ cts? Yes!