$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, "f_i"'] & \prod_{k=1}^{n} X_k \arrow[d, "\pi_i"] \\ & X_i \end{tikzcd}$$

$X_1 \times X_2 \times \cdots \times X_n$

π_i:

Characteristic property of product topology.

$$\boxed{f \text{ is continuous iff each } f_i \text{ is continuous}}$$

$$f(x,y) = (x^2 - y^2,\ xy)$$

$$f_1(x,y) = x^2 - y^2$$

$$f_2(x,y) = xy$$

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, "f_i"'] & \prod_{k=1}^{n} X_k \arrow[d, "\pi_i"] \\ & X_i \end{tikzcd}$$

Claim: f is continuous iff each f_i is.

Suppose f is continuous. Then $f_i = \pi_i \circ f$ is continuous as it is a composition of continuous functions.

(Each π_i is continuous by construction).

Conversely, suppose each f_i is continuous.

To show f is continuous we show $f^{-1}(S)$ is open

For every subbasic open set $\pi_j^{-1}(V)$ where $V \subseteq X_j$ is open.

$$B = S_1 \cap \cdots \cap S_k$$

$$f^{-1}(B) = f^{-1}\left(\bigcap_{j=1}^{k} S_j\right) = \bigcap_{j=1}^{k} f^{-1}(S_j)$$

$S = \pi_j^{-1}(\underset{\substack{\uparrow \\ \text{open}}}{V})$

$$\begin{aligned} f^{-1}(S) &= f^{-1}(\pi_j^{-1}(V)) \\ &= (\pi_j \circ f)^{-1}(V) \\ &= f_j^{-1}(V) \leftarrow \text{open since } f_j \text{ is continuous.} \end{aligned}$$

τ generated by subbasis $\mathcal{S}$

on X

$f : Z \to X$

f is cts iff $\quad f^{-1}(S)$ is open for all $S \in \mathcal{S}$

$\pi_2^{-1}(V) = \{ x \in X_1 \times \cdots \times X_{18} : \pi_2(x) \in V \}$

The characteristic property of the prod top. is characteristic!

CPPT

Suppose τ is a topology on $X = \prod_{k=1}^{n} X_k$

satisfying whenever $f: Z \to X$ is a map

f is continuous iff $\pi_j \circ f := f_j$ is for each j.

Then τ is the product topology.

Let (X, τ) be $X_{(r)}$, r for random

and let $X_{(p)}$ be X with the product topology

I claim that each $\pi_j^{(r)} : X_{(r)} \to X_j$ is continuous.

ι_Δ

$\Delta = X$

$X \quad X$

$$\begin{tikzcd} X_{(r)} \arrow[r, "Id_{r,n}"] \arrow[dr, "\pi_j^{(r)}"'] & X_{(r)} \arrow[d, "\pi_j^{(n)}"] \\ & X_j \end{tikzcd}$$

$$X \xrightarrow{g} X$$

$$g(x) = x$$

Since $Id_{r,n}$ is continuous the CPPT implies each $\pi_j^{(n)}$ is.

$$\begin{tikzcd} X_{(r)} \arrow[r, "Id_{r,p}"] \arrow[dr, "\pi_j^{(r)}"'] & X_{(p)} \arrow[d, "\pi_j^{(p)}"] \\ & X_j \end{tikzcd}$$

Since each $\pi_j^{(r)}$ is continuous so is $Id_{r,p}$ by the CPPT.

The same diagram with p and r interchanged shows

$Id_{p,r}$ is also continuous, so is a homeomorphism.

Facts: 1) A product of two Hausdorff spaces is Hausdorff

Exercise.

$$\underbrace{(X_1 \times X_2) \times X_3 \sim X_1 \times X_2 \times X_3}$$

Homework CAPT!

2) If $\mathcal{B}_1$ is a basis for X_1 and $\mathcal{B}_2$ is a basis for X_2 then $\mathcal{B} = \{ B_1 \times B_2 : B_1 \in \mathcal{B}_1 \; B_2 \in \mathcal{B}_2 \}$ is a basis for $X_1 \times X_2$.

Exercise!

$U \subseteq X_1 \times X_2$, open

given $x \in U$ $\exists$ $B \in \mathcal{B}$ $x \in B \subseteq U$.

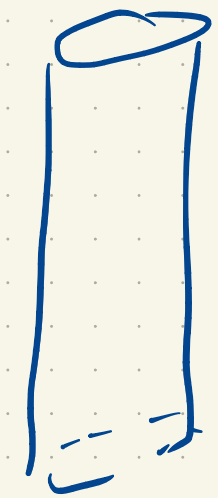
$x \in \underbrace{V \times W} \subseteq U$

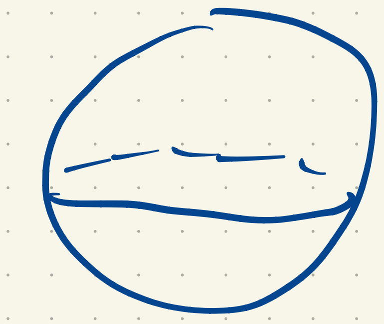
3) A product of two second countable spaces is second countable.

Is a product of two manifolds again a manifold?

Yes! $M^{d_1} \times M^{d_2} \to M^{d_1 + d_2}$

$S^1 \times \mathbb{R}$





$\mathbb{R}^2 \times \mathbb{R}^1$

$S^1 \times S^1 = \mathbb{T}^2$ torus

$\underbrace{S^1 \times \cdots \times S^1}_{k} = \mathbb{T}^k$

Products

Quotients