Previously on Math 651....

- Second countable spaces admit a countable basis

  $\Rightarrow$ first countable $\Rightarrow$ sequence arguments

  $\Rightarrow$ separable (admit a countable dense subset)

  $\Rightarrow$ Lindelöff (every cover has a countable subcover)

- Manifolds
  - Locally Euclidean of dimension $n$ (each point has a nbhd $\sim \mathbb{R}^n$)
  - Hausdorff
  - $2^{nd}$ countable

  (Many examples asserted, but few proofs)

- Subspace Topology

$A \subseteq X \qquad \tau_A = \{ U \cap A : U \text{ is open in } X \}$

- If $X$ is Hausdorff so is $A$
- If $X$ is 2nd countable so is $A$
- If $X$ is a metric space, the subspace top + metric top on $A$ agree
- $i_A : A \to X$ is always continuous

  In fact: if $\hat{\tau}$ is any topology on $A$ and $i_A$ is cts $(A, \hat{\tau}) \to X$ then $\tau_A \subseteq \hat{\tau}$. It is the coarsest topology for which $i_A$ is continuous.

# Big Deal: Characteristic Property of Subspace Topology:

$$\begin{tikzcd} & X \\ Z \arrow[r, "f"'] \arrow[ur, "i_A \circ f"] & A \arrow[u, "i"'] \end{tikzcd}$$

The function $f$ is continuous iff $i_A \circ f$ is.

"A function is continuous into a subspace iff it is continuous into the ambient space."

Two easy facts

1) If $f: X \to Z$ is cts and $A \subseteq X$

Then $f|_A : A \to Z$ is cts.

$Z$

$f|_A = f \circ i_A$ (restriction of domain)

2) If $f: Z \to X$ is cts but $f(Z) \subseteq Y$ then

$\hat{f}: Z \to Y$ is cts (restriction of codomain)

$$\begin{tikzcd} & X \\ Z \arrow[ur, "f"] \arrow[r, "\hat{f}"'] & Y \arrow[u, hook, "i_Y"'] \end{tikzcd}$$

$f$ is cts, so $\hat{f}$ is.

(We typically don't decorate...)

What do I mean by "characteristic property"? It's a property that <u>defines</u> the topology abstractly.

We say a topology on $A \subseteq X$ satisfies the char property if whenever $f: Z \to A$ is a map, then $f$ is cts iff $\tilde{f} = f \circ i_A$ is.

$$\begin{tikzcd} & X \\ Z \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & A \arrow[u, "i_A"'] \end{tikzcd}$$

Claim: A topology on $A$ that satisfies the characteristic property of the subspace top is the subspace top.

Note:

Pf: Let $A_s$ be $A$ with the subspace top and let $A_r$ be $A$ with a random topology satisfying the char prop.

I want to show $Id_{sr}: A_s \to A_r$

$Id_{rs}: A_r \to A_r$ are continuous, in which case the topologies are identical! $\big(id: (Z,\tau_1) \to (Z,\tau_2)$ is cts iff $\tau_1 \supseteq \tau_2\big)$

① 
$$\begin{tikzcd} & X \\ A_r \arrow[ur, "\widetilde{Id}_{r,r}"] \arrow[r, "Id_{r,r}"'] & A_r \arrow[u, hook, "i_{A_r}"'] \end{tikzcd}$$

Since $Id_{r,r}$ is cts $\Rightarrow$ $\widetilde{Id}_{r,r}: A_r \to X$ is cts

$\Rightarrow$ $i_{A_r}: A_r \to X$ is cts.

(2)

$$\begin{tikzcd} & X \\ A_r \arrow[ur, hook, "i_{A_r}"] \arrow[r, "\mathrm{Id}_{rs}"'] & A_s \arrow[u, hook, "i_s"'] \end{tikzcd}$$

Since $i_{A_r}$ is cts so is $\mathrm{Id}_{rs}$

(3)

$$\begin{tikzcd} & X \\ A_s \arrow[ur, hook, "i_{A_s}"] \arrow[r, "\mathrm{Id}_{sr}"'] & A_r \arrow[u, hook, "i_{A_r}"'] \end{tikzcd}$$

Since $i_{A_s}$ is cts so is $\mathrm{Id}_{sr}$.

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Prop: Suppose $X$ is a top space and $A \subseteq B \subseteq X$.

Then the subspace topologies on $A$ as subspaces of $B$ and $X$ coincide.

Pf: Let $A_B$ and $A_X$ denote the two topolgies

We'll show that $A_B$ satisfies the char property

$$\begin{tikzcd}
 & X \\
 & B \arrow[u, hook, "i_B"'] \\
Z \arrow[r, "f"'] \arrow[ur, "\tilde{f}"] \arrow[uur, bend left, "\tilde{\tilde{f}}"] & A_B \arrow[u, hook, "i_{A_B}"']
\end{tikzcd}$$

$f$ is cts $\Leftrightarrow$ $\tilde{f}$ is by char property applied to $A \hookrightarrow B$.

$\tilde{\tilde{f}}$ is cts iff $\tilde{f}$ is by char property $B \hookrightarrow X$.

So $\tilde{\tilde{f}}$ is cts iff $f$ is.

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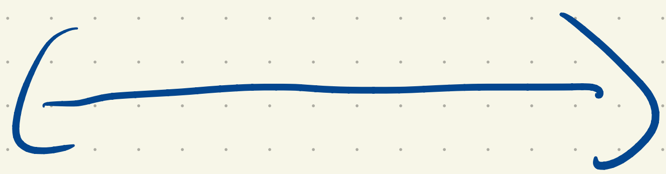
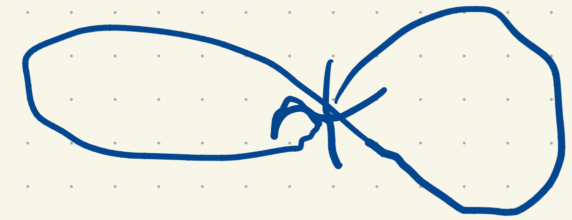
Def A map $f: X \to Y$ is a top embedding if it $f$ is a homeomorphism onto its image. (w/ subspace)

$\Rightarrow$ injective cts!

not a homeo.





Intuitively not a homeo.

Super common construction.

But first:

1) $x_j \to x$ in $\mathbb{R}^n$
   $y_j \to y$ in $\mathbb{R}^k$ $\Rightarrow$ $(x_j, y_j) \to (x, y)$ in $\mathbb{R}^{n+k}$

2) $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is cts.

(Exercises)

$A \subseteq \mathbb{R}^n$ subspace

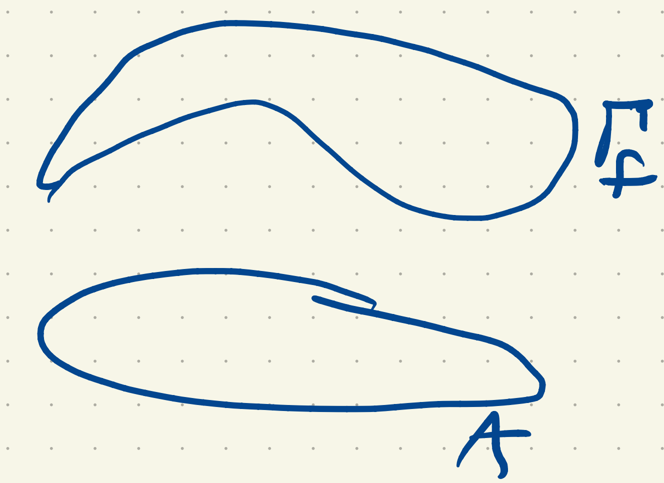
$f: A \to \mathbb{R}^k$

Graph of $f$ $\quad \Gamma_f = \{ (x, f(x)) : x \in A \}$.

Then $\Phi: A \to \Gamma_f$ $\quad x \longmapsto (x, f(x))$ is a topological embedding

Pf: $\Phi(x)$ is injective (obvious)

cts (metric space arguments)




Inverse is $\pi: \mathbb{R}^{n+k} \to \mathbb{R}^k$ which is cts.

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Claim: $S^n$ is a manifold. $\quad S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \}$

2nd count, Hausdorff trivial

$S^n_+ = \{x \in S^n : x_{n+1} > 0\}$

$U = \pi_{n+1}^{-1}((0,\infty))$ so is open.

$S^n_+ = U \cap S^n$ so is open in $S^n$.

$f : B \to \mathbb{R} \quad x \to \sqrt{1-|x|^2}$ is cts (calc III)

$S^n_+$ is $\Gamma_f$ so is homeomorphic (as a subspace of $\mathbb{R}^{n+1}$ and hence about $S^n$) to $B$.

$S^n_- = \{x \in S^n : x_{n+1} < 0\}$ is homeomorphic to $S^n_+$

via $R(x, x_{n+1}) \to (x, -x_{n+1})$

which is cts $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and hence

also $S^{n+1} \to \mathbb{R}^{n+1}$

and $S^{n+1} \to S^{n+1}$.

Most points in $S^n$ now covered. (only those w/ $x_{n+1} \neq 0$)

Exercise: $(x_1, x_2, \ldots, x_n, x_{n+1}) \to (x_1, x_2, \ldots, x_{n+1}, x_n)$

descends to homeomorphism $S^n \to S^n$