e.g: 1) $\mathbb{R}$, $\mathcal{B} = \{(a,b) : a < b\}$ (these are exactly balls)

2) $\mathbb{R}$ $\mathcal{B}' = \{(a,b) : a<b, a,b \in \mathbb{Q}\}$.

$\mathcal{B}$ is contained in the standard topology
If U is open and $p \in U$, is a,b
$p \in (a,b) \subseteq U$.

Is $a',b' \in \mathbb{Q}$, $p \in (a',b') \subseteq (a,b) \subseteq U$.

2') Alt. HW: $\tau_{\mathcal{B}}$ is the smallest top that includes $\mathcal{B}$.

Evidently $\mathcal{B}'$ is a prebasis. $\mathcal{B}' \subseteq \mathcal{B} \subseteq \tau_{\mathcal{B}}$
$\Rightarrow \tau_{\mathcal{B}'} \subseteq \tau_{\mathcal{B}}$.

Each $(a,b) \in \mathcal{B}$ is a union of elements of $\mathcal{B}'$.
So $\mathcal{B} \subseteq \tau_{\mathcal{B}'}$. So $\tau_{\mathcal{B}} \subseteq \tau_{\mathcal{B}'}$.

3) X any set $\mathcal{B} = \{\{x\} : x \in X\}$.

Prebasis?
$\tau_{\mathcal{B}}$ is discrete.

open in discrete? yup
every open set a union? yup.

4) X any set $\mathcal{B} = \{X\}$.
Prebasis?
$\tau_{\mathcal{B}}$ is indiscrete.

open in ind.
every open set a union?

5) $\mathbb{R}$, $\mathcal{B} = \{ [c,d) : c < d \}$

Prebasis? $\mathbb{R} = \bigcup [-n, n)$.

If $B_1, B_2 \in \mathcal{B}$ then $B_1 \cap B_2 \in \mathcal{B}$ or is empty. ✓

So $\mathcal{B}$ generates $\tau_{\mathcal{B}}$. Note, $[0,1)$ is in $\tau_{\mathcal{B}}$

but not in $\tau_{\mathbb{R}}$. So $\tau_{\mathcal{B}} \not\subseteq \tau_{\mathbb{R}}$.

But (a,b) is a union of sets $[a_n, b)$

So $\tau_{\mathbb{R}} \underset{\underset{\neq}{\uparrow}}{\subseteq} \tau_{\mathcal{B}}$. This is a strictly finer

topology. (Lower limit topology $\mathbb{R}_\ell$).

$$\tau_{\mathbb{R}} \subseteq \tau_{\mathbb{R}_\ell}.$$

More open sets $\Rightarrow$ harder to converge.

$x_n = -\frac{1}{n}$ $\qquad x_n \to 0$ in $\tau_{\mathbb{R}}$

$\qquad x_n \not\to 0$ in $\tau_{\mathbb{R}_\ell}$

($[0,1)$ is an open set about 0 that excludes the entire sequence.)

Def: A neighbourhood base at $x \in X$ is a subset $\mathcal{W} \subseteq \mathcal{V}(x)$ such that

for all $U \in \mathcal{V}(x)$, $\exists$ $W \in \mathcal{W}$, $x \in W \subseteq U$.

(In particular, $\mathcal{V}(x)$ is a nbhd base; but its not the only one. Ignore previous def!)

($\mathcal{V}(x) \to$ neighbourhoods of x $\quad \mathcal{V} \to$ voisin)

Def: A space X is <u>first countable</u> if each $x \in X$ admits a countable nbhd base.

E.g. metric spaces are 1^{st} countable. $\{B_r(x) : r \in \mathbb{Q}, r > 0\}$

Indeed, it's hard to find examples that are not (HW).

Def: A nbhd base $\{W_k\}_{k=1}^{\infty}$ is nested if

$W_k \subseteq W_j$ whenever $k \geq j$.

Lemma: If $x \in X$ admits a countable nbhd base, it admits a nested countable nbhd base.

Pf:

Pf: Let $W'_k = \bigcap_{j=1}^{k} W_j$.

In first countable spaces, we can often detect top properties by sequences.

E.g.

Prop: Let X be 1^{st} countable and let $V \subseteq X$. Then p is a contact point iff there is a seq in V converging to p.

Pf: (Nontrivial direction).

Suppose p is a contact point and let $\{W_k\}_{k=1}^{\infty}$ be a nested nbhd base at p.

For each k, pick $p_k \in W_k \cap V$.

Let U be open about p and find W_K, $W_K \subseteq U$.

If $k \geq K$, $p_k \in W_k \subseteq W_K \subseteq U$. So $p_k \to p$.

See also Prop 2.48 (We effectively just proved it).

We had noted that Hausdorfness ensures richness

We will often restrict richness via the size of a relevant basis.

Def: A space is 2^{nd} countable if it admits a countable basis.

E.g $\mathbb{R}$ $\left((a,b) \quad a,b \in \mathbb{Q} \right)$

Exercise: $\mathbb{R}^n$ is 2^{nd} countable. (rational radii, rational coords)

Exercise: $\mathbb{R}$, discrete. not 2^{nd} countable.

Challenge: $\mathbb{R}_\ell$: 2^{nd} countable or not.

Evidently 2^{nd} countable $\Rightarrow$ 1^{st} countable, but not vice-versa. Discuss!

It's hard to motivate the value of 2^{nd} countability, but here's an example.

Prop: If X is 2^{nd} countable it admits a countable dense subset.

Pf: Let $\{B_k\}$ be a countable basis.
For each k, pick $p_k \in B_k$. I claim $V = \{p_k\}$ is dense. Indeed let $p \in X$, then $\exists\ k,\ p \in B_k \subseteq U$. $U \in \mathcal{U}(p)$.

Since $p_k \in B_k \subseteq U$, $U \cap V \neq \emptyset$.

Remark: A space admitting a countable dense subset is called <u>separable</u>.

Def Let $A \subseteq X$. An <u>open cover</u> of A is a collection $\mathcal{G}$ of open sets such that $A \subseteq \bigcup_{G \in \mathcal{G}} G$. A subcover of $\mathcal{G}$ is a subset $\mathcal{G}' \subseteq \mathcal{G}$ that is still an open cover.

Def: A space is Lindelöf if every open cover admits a countable subcover. (kind of smallness)

Prop: 2^{nd} countable spaces are Lindelöf.

Pf: Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover.

Let $\mathcal{B}$ be a countable basis and let $\mathcal{B}' \subseteq \mathcal{B}$ be those elements that are contained in a set U_α.

For each $B \in \mathcal{B}'$, pick α_B such that $U_{\alpha_B} \supseteq B$.

I claim $\{U_{\alpha_B}\}_{B \in \mathcal{B}'}$ is a cover of X.

Let $x \in X$. So $x \in U_\alpha$ for some α. So $\exists B$, $x \in B \in U_\alpha$ But then $x \in B_{\alpha_B}$.

Manifolds:

Def: A space X is <u>locally Euclidean</u> of dimension n if each $p \in X$ has a nbhd that is homeomorphic to an open set in $\mathbb{R}^n$.

Exercise: equivalently: homeo to $B_1(0) \subseteq \mathbb{R}^n$
homeo to $\mathbb{R}^n$

$$\left(B_1(0) \sim \mathbb{R}^n \, ! \right)$$

One might be tempted to study loc. euc. spaces, but there are pathologies.

There are loc Euc spaces that are not Hausdorff.

Exercise: Every loc Euc space is 1st countable.

But it turns out that loc Euc spaces need not have a countable dense subset.

Exercise: A loc Euc space is separable iff it is 2^{nd} countable.

Def: A manifold of dimension n is a topological space that is

a) Loc euc of dimension n

b) Hausdorff

c) 2^{nd} countable.

ballanced of rich + fine.

Prop: If $U \subseteq \mathbb{R}^n$ is open, it is an n-manifold.

Pf: Evidently loc. Euc.

Hausdorff: $V, W \to V' = V \cap U \quad W' = W \cap U.$

countable basis: $\mathcal{B}' = \{ B \in \mathcal{B} : B \subseteq U \}.$

open in U? Yup!

If $W \subseteq U$ is open, it is a union of elements of $\mathcal{B}$ and hence $\mathcal{B}'$.