Last class:

Defined continuity, and gave examples.

$f : X \to Y$ is cts if $f^{-1}(U)$ is open in X whenever U is open in Y.

We'll be working with preimages a lot, so the following facts are useful:

$$f^{-1}\Big(\bigcup_{\alpha \in I} A_\alpha\Big) = \bigcup_{\alpha \in I} f^{-1}(A_\alpha)$$

$$f^{-1}\Big(\bigcap_{\alpha \in I} A_\alpha\Big) = \bigcap_{\alpha \in I} f^{-1}(A_\alpha)$$

$$f^{-1}(A^c) = f^{-1}(A)^c$$

The forward version is trickier

$$f\Big(\bigcup_{\alpha \in I} A_\alpha\Big) = \bigcup_{\alpha \in I} f(A_\alpha)$$

but

$$f\Big(\bigcap_{\alpha \in I} A_\alpha\Big) \neq \bigcap_{\alpha \in I} f(A_\alpha)$$

$$f(A^c) \neq f(A)^c$$

Exercise:

The first is a contrapositive, and the second is also with an surjectivity hypotheses

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Now suppose $U \subseteq X$ is open.

Then U inherits a natural topology

$$\tau_U = \{ V : V \subseteq U, V \in \tau \}$$

Easy to see this is a topology.

If $f : X \longrightarrow Y$ is continuous and $U \subseteq X$ is open

then $f|_U : U \to Y$ is also continuous.

Indeed, if $W \subseteq Y$ is open,

$$f|_U^{-1}(W) = f^{-1}(W) \cap U$$

which is open $\boxed{\text{in } U.}$

important words.

We have a strong converse, effectively that continuity is local: If each $p \in X$ has a nbhd on which f is cts, then f is cts.

Prop: Suppose $f: X \to Y$ and for each $p \in X$ there exists an open set U_p containing p such that $f|_{U_p}$ is continuous. Then f is continuous.

Pf: Let $W \subseteq Y$ be open. Then

$$\begin{aligned} f^{-1}(W) &= X \cap f^{-1}(W) \\ &= \left(\bigcup_{p \in X} U_p \right) \cap f^{-1}(W) \\ &= \bigcup_{p \in X} \left(U_p \cap f^{-1}(W) \right) \\ &= \bigcup_{p \in X} f|_{U_p}^{-1}(W). \end{aligned}$$

Each $f^{-1}\big|_{U_p}^{-1}$ is open in O_p and hence also open in X. So $f^{-1}(W)$ is a union of open sets in X and is open

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Topological spaces admit a notion of sameness.

Def: $f : X \to Y$ is a homeomorphism if

1) it is a bijection
2) it is continuous
3) its inverse is also continuous.

For example: $\mathbb{R}$ is homeomorphic to $(-\frac{\pi}{2}, \frac{\pi}{2})$

$f(x) = \arctan(x)$ is a bijection and cts

$f^{-1}(x) : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is $\tan(x)$, also cts.

In fact homeomorphism yields an equivalence relation on top spaces: $X \sim Y$ and $Y \sim Z \Rightarrow X \sim Z$, etc.

It might be hard to visualize the interesting mode of failure:

The continuity of the inverse.

e.g. $f: [0, 1) \to S^1 = \{ z \in \mathbb{R}^2 : |z| = 1 \}$

(metric spaces)

$f(x) = (\cos(2\pi x), \sin(2\pi x))$ is continuous.

(Why?)

And it's bijective.

But on your homework you show its inverse is not continuous.

Now just because <u>this</u> f didn't work we can't claim that <u>none</u> exist. But later in the class we'll be able to show those spaces are not cts.

Def: Given two top spaces τ_1 and τ_2, we say

τ_1 is <u>finer</u> than τ_2 if $\tau_1 \supseteq \tau_2$.

The finer a topology, the easier it is for

$$f : (X, \tau_1) \to Y \quad \text{to be cts.}$$

The coarser, the easier it is for

$$g : Y \to (X, \tau_2) \quad \text{to be cts.}$$

In fact for any top space Y,

$$X_{disc} \xrightarrow{f} Y \xrightarrow{g} X_{ind} \quad \text{are always cts.}$$

However: If X has more than one element then

$f : \mathbb{R} \to X_{dis}$ and

$g : X_{ind} \to \mathbb{R}$ are cts $\Leftrightarrow$

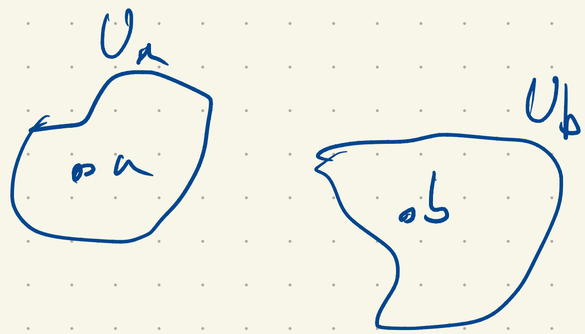
they are constant

(challenge!)

Anyway: good top spaces strike a balance between being too fine and too coarse.

To prevent being too coarse:

Def: A topological space is Hausdorff if for all $a, b \in X$ there exist nbhds U_a, U_b of a, b with $U_a \cap U_b = \emptyset$.



(Singletons are far apart)

If X has more than one point, X_{ind} fails (spectacularly) to be Hausdorff.

Every metric space is Hausdorff:

Cor: Every metrizable space is Hausdorff

Cor: An indiscrete space with more than one point is not metrizable.

Convergence of sequences.

A sequence $\{x_n\}$ in X <u>converges</u> to x if

For every open set U containing x there exists N

s.t. if $n \geq N$, $x_n \in U$.

Exercise: If X is a metric space, this is the usual notion of convergence.

Prop: In a Hausdorff space limits are unique.

Pf: Suppose $x_n \to y$ and $z \neq y$.

Find U, V, $U \cap V = \emptyset$, $y \in U$, $z \in V$.

Pick N, $n \geq N \Rightarrow x_n \in U$. Then for all $n \geq N$, $x_n \notin V$.

So $x_n \not\to z$ (if $x_n \to z$ then inf many terms are in V).

Prop: In Hausdorff spaces singletons and finite sets are closed.

(This is a weaker property, T_1) (Hausdorff is T_2)