Observe: the boundary of A is closed.

There is another w to express this.

$$\overline{A}^{c} = \mathrm{Ext}(A) \Rightarrow \overline{A} = (\mathrm{Ext}\, A)^{c}$$

$$\overline{A^{c}}^{c} = \mathrm{Ext}(A^{c}) \Rightarrow \overline{A^{c}} = \mathrm{Ext}(A^{c})^{c}$$

So: $\partial A = \overline{A} \cap \overline{A^{c}}$

$$= \mathrm{Ext}(A)^{c} \cap \mathrm{Ext}(A^{c})^{c}$$

$$= \left[\mathrm{Ext}(A) \cup \mathrm{Ext}(A^{c})\right]^{c}$$

$$= X \setminus \left(\mathrm{Ext}(A) \cup \mathrm{Ext}(A^{c})\right)$$

Moreover: $x \in \mathrm{Ext}(A^{c}) \Leftrightarrow \exists\, U \in \mathcal{V}(x),\ U \subseteq (A^{c})^{c}$

$\Leftrightarrow \exists\, U \in \mathcal{V}(x)\ \ U \subseteq A$

$\Leftrightarrow x \in \mathrm{Int}(A).$

So $\partial A = X \setminus \left(\mathrm{Ext}(A) \cup \mathrm{Int}(A)\right).$

What is the boundary of $\mathbb{Q} \in \mathbb{R}$? $\mathbb{Q}$.

(Every point of $\mathbb{R}$ is a contact point of $\mathbb{Q}$ and $\mathbb{Q}^c$.)

Text: prop 2.8 contains a number of related inter-relationships, which are left as exercises. You must prove these before using.

Def $x \in X$ is a <u>limit point</u> of $A \subseteq X$ if for every $U \in \mathcal{U}(x)$, $U \cap A$ contains a point aside from x.

(Note x may or may not be in A)

Exercise: Every limit point of A is a contact point of A.

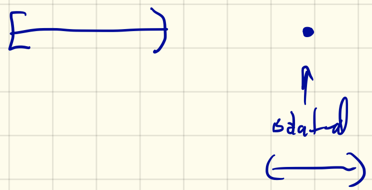
If $x \notin A$, x is a limit point of A $\Leftrightarrow$ x is a contact point of A.

What's left over? What are contact points that are not limit points. Suppose x is one of these.

$x \in A$ then. And $\exists$ $U \in \mathcal{U}(x)$ $U \cap A = \{x\}$.

Def: We say $x \in A$ is an *isolated* point of A if $\exists\, U \in \mathcal{V}(x)$, $U \cap A = \{x\}$.

Exercise: The set of contact points of A is the disjoint union of the limit points of A and the isolated points of A.



Exercise: A set A is closed iff it contains its limit points.

(hint contact points not in A must be limit points)

Def:

A set $A \subseteq X$ is <u>dense</u> in X if $\overline{A} = X$.

E.g. $\mathbb{Q} \subseteq \mathbb{R}$

Every point of X is a contact point of A.

Every point of X is adjacent to the points of A.

(A is near everything)

Continuity.

Recall

Def: Let $f: X \to Y$ be a map between metric spaces.

Then f is continuous if whenever $x_n \to x$ in X,

$f(x_n) \to f(x) \in Y$.

Prop: $f: X \to Y$ is cts $\iff$ it is cts'

Pf: Suppose cts' and suppose $p_n \to p$ in X.

Let $\varepsilon > 0$. Find δ so $f(B_\delta(p)) \subseteq B_\varepsilon(f(p))$.

Find N so $n \geq N \Rightarrow p_n \in B_\delta(p)$.

Then if $n \geq N$, $f(p_n) \in f(B_\delta(p)) \subseteq B_\varepsilon(f(p))$.

So $f(p_n) \to f(p)$.

Now suppose not cts'. So there exists $p \in X$, and $\varepsilon > 0$ s.t. for all $\delta > 0$, $f(B_\delta(p)) \not\subseteq B_\varepsilon(f(p))$.

For each n, pick $p_n \in B_{1/n}(x)$, $f(p_n) \notin B_\varepsilon(f(p))$.

So $p_n \to p$ but so $f(p_n) \not\to f(p)$.

So not cts.

Third characterization.

Def: $f: X \to Y$ is cts'' if whenever $U \subseteq Y$ is open $f^{-1}(U) \subseteq X$ is open.

Recall: $f^{-1}(W) = \{x : f(x) \subset W\}$.

$$f(A) \subseteq W \iff A \subseteq f^{-1}(W)$$

Prop: cts' $\iff$ cts''

Pf: Suppose cts'. Let $U \subseteq Y$ be open.

Let $p \in f^{-1}(U)$. Pick $\varepsilon > 0$ with $B_\varepsilon(f(p)) \subseteq U$.

Pick δ with $f(B_\delta(p)) \subseteq B_\varepsilon(f(p))$.

So $B_\delta(p) \subseteq f^{-1}(B_\varepsilon(f(p))) \subseteq f^{-1}(U)$. So $f^{-1}(U)$ is open.

Now suppose cts''. Let $p \in X$ and consider $U = B_\varepsilon(f(p))$, which is open. Now $p \in f^{-1}(U)$, which is open in X. So there exists $\delta > 0$ and $B_\delta(p) \subseteq f^{-1}(U)$.

So f is cts'.

So there is a characterization of continuity for metric spaces solely in terms of open sets (cts").

Def: Let $f: X \to Y$ be a map between top spaces. Then f is <u>continuous</u> if for every U open in Y, $f^{-1}(U)$ is open in X.

E.g.: 1) Every map that was continuous before you knew the definition of topology.

2) Constant functions. $\left(f^{-1}(U) = \begin{cases} X \\ \phi \end{cases} \right)$

3) $Id: X \to X$ $\quad (f^{-1}(U) = U)$.

4) A composition of cts functions

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$\uparrow$$

$$U$$

$$(f \circ g)^{-1}(U) = \{x \in X, g(f(x)) \in U\}$$

$$= \{x \in X : f(x) \in g^{-1}(U)\}$$

$$= \{x \in X; \ x \in f^{-1}(g^{-1}(U))\}$$

$$= \underbrace{f^{-1}(g^{-1}(U))}_{\text{open, open}}.$$