Motivating Theorem

Suppose d_1 and d_2 are two metrics on X.
Then TFAE

1) For all sequences $\{x_n\}$, if $x_n \xrightarrow[d_2]{} x$, then $x_n \xrightarrow[d_1]{} x$,

2) For all functions $f: X \to \mathbb{R}$, if f is cts w.r.t. d_1 then f is cts w.r.t. d_2

3) For all $U \subseteq X$, if U is open w.r.t. d_1 then U is open w.r.t. d_2

4) For all $V \subseteq X$, if V is closed w.r.t. d_1 then U is closed w.r.t. d_2.

In particular: two metrics determine the same convergent seq's

$\Leftrightarrow$

determine same cts $X \to \mathbb{R}$

$\Leftrightarrow$ same open sets

$\Leftrightarrow$ same closed sets.

One might hope to add $\Leftrightarrow$ they are equivalent, in which case the right object of study might be equivalence classes of metrics. But no.

$$d'(x,y) = \left| \int_x^y e^s \, ds \right| = \left| e^y - e^x \right|$$

Exercise: Show d' is a metric, but not equivalent to the standard metric.

We'll shortly have a good tool for seeing that this metric generates the same open sets as the standard metric, though.

So we're going to dump the notion of metric entirely, and use property (3) as the foundation.

Def: Let X be a set. A <u>topology</u> on X is a collection τ of subsets of X satisfying

1) $\tau \supseteq \{X, \emptyset\}$

2) If $\{U_\alpha\}_{\alpha \in I} \subseteq \tau$, $\bigcup_{\alpha \in I} U_\alpha \in \tau$

3) If $\{U_k\}_{k=1}^{n} \subseteq \tau$ $\bigcap_{k=1}^{n} U_k \in \tau$

We call the elements of τ the open sets of the top.

We call (X, τ) a topological space (and drop τ when it is implicit).

We should verify that the open sets of a metric space form a topology.

1) ϕ and X are open.

2) Suppose $\{U_\alpha\}_{\alpha\in I}$ is a family of open sets.

Let $p \in \bigcup U_\alpha$. So $\exists$ $\alpha' \in I$ with $p \in U_{\alpha'}$.

Since $U_{\alpha'}$ is open, there exists $r>0$ s.t. $B_r(p) \subseteq U_{\alpha'}$
$\subseteq \bigcup U_\alpha$.

3) Suppose $U_1, \dots, U_n$ are open. Let $p \in \bigcap U_k$.

So for each k, $\exists$ r_k, $B_{r_k}(p) \subseteq U_k$.

Let $r = \min(r_1, \dots, r_n)$. Then $B_r(p) \subseteq B_{r_k}(p) \subseteq U_k$

for each k and $B_r(p) \subseteq \bigcap U_k$.

Every set has two important, natural, and uninteresting topology.

1) The discrete topology: $\tau = \mathcal{P}(X)$.

Singletons are open sets!

(Easy to verify this is a top)

2) The indiscrete topology $\tau = \{X, \phi\}$

Trivial to verify this is a topology.

Sometimes called the trivial top.

Exercise: Show that the discrete metric on X generates the discrete topology.

On the other hand, suppose $X = \{a, b\}$ and give X the indiscrete top. Does this arise from a metric on X?

No: Let d be a metric.

Let $r = d(a,b)$.

Then $B_{r/2}(a) = \{a\} \longrightarrow$ not in the indiscrete top.

So the study of topologies is strictly broader than the study of metric spaces.

We say a space is <u>metrizable</u> if there is a metric that generates the topology (in which case there is more than one choice).

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Def: A set $V \subseteq X$ is closed if $V^c \ (= X \setminus V)$ is open.

Exercise: a) X, ϕ are cl ∩

Recall deMorgan's Laws:
$$\left(\bigcup_{\alpha \in I} A_\alpha\right)^c = \bigcap_{\alpha \in I} A_\alpha^c$$

$$\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} A_\alpha^c$$

(in negation, $\forall \to \exists$, $\exists \to \forall$)

Exercise: Use deMorgan's Laws to prove that

1) An arbitrary intersection of closed sets is closed

2) A finite union of closed sets is closed.

E.g. In a metric space, $\overline{B_r}(x) = \{y : d(x,y) \leq r\}$

Exercise: $\overline{B_r}(x)$ is closed. (Use Δ ineq!)

We call $\overline{B_r}$ the <u>closed</u> ball of radius r.

In $\mathbb{R}$,
$$B_r(x) = (x-r, x+r)$$
$$\overline{B_r}(x) = [x-r, x+r].$$

A set is a pile of objects without structure.

A topology on a set encodes a notion of adjacency or nearness.

To formalize this we introduce

Def Let $A \subseteq X$, a top space.

The <u>interior</u> of A, $\mathrm{Int}(A)$ is the union of all open sets contained in A.

The closure of A, $\overline{A}$ is the intersection of all closed sets containing A.

Evidently, the interior of a set is open, and the exterior is closed.

Exercise: The <u>interior</u> of a set is the largest open set it contains. The closure of a set is the smallest closed set that contains it.

Exercise: A set is open iff $A = \mathrm{Int}\, A$.
A set is closed iff $A = \overline{A}$.

Def: A point $x \in X$ is a contact point of $A \subseteq X$ if every open set U containing x satisfies $U \cap A \neq \emptyset$.

Note x may or may not be in A.

E.g. $X = \mathbb{R}$ $A = (-1,1)$. The set of contact points is $[-1,1]$.

$A = \mathbb{Q}$. The set of contact points is $\mathbb{R}$,

$$\left((x-\varepsilon, x+\varepsilon) \cap \mathbb{Q} \neq \emptyset \ \forall \varepsilon > 0 \right)$$

Prop: $\bar{A}$ is the union of contact points of A.

Pf: Let A' denote the set of contact points.

Consider $q \in \bar{A}^c$. There is an open set U containing q such that $U \cap \bar{A} = \emptyset$. Hence $U \cap A = \emptyset$ and q is not a contact point. I.e. $\bar{A}^c \subseteq (A')^c$ and therefore $A' \subseteq \bar{A}$.

Now suppose $x \notin A'$. Then there is an open set U with $x \in U \subseteq A^c$. Let $V = U^c$, so V is closed and $A \subseteq V$. Hence $\bar{A} \subseteq V$. Since $x \notin V$, $x \notin \bar{A}$. That is $(A')^c \subseteq (\bar{A})^c$ and $\bar{A} \subseteq A'$.

The contact points of A are the points in or adjacent to A.

Def The *exterior* of A, $\mathrm{Ext}(A)$, is $X \setminus \bar{A} = \bar{A}^c$.

That is, $x \in \mathrm{Ext}\, A \iff x$ is not a contact pt,

$$\iff \exists\, U \in \mathcal{T},\ x \in U,\ U \cap A = \emptyset.$$

These are the points not adjacent to A.

This notion of all the open sets containing a point shows up frequently.

Def: Let $x \in X$. A neighborhood of x is an open set containing x. The set of all open sets containing x, the *neighbourhood base* of x is denoted $\mathcal{U}(x)$.

What is a point that is adjacent both to A and to A^c?

These points are in $\bar{A}$ and in $\overline{A^c}$.

Def: The *boundary* of A is $\bar{A} \cap \overline{A^c}$.

Exercise $x \in \partial A \iff \forall\, U \in \mathcal{U}(x),\ U \cap A \neq \emptyset$
$U \cap A^c \neq \emptyset$.

Observe: the boundary of A is closed.

There is another w to express this.

$$\overline{A}^c = \mathrm{Ext}(A) \Rightarrow \overline{A} = (\mathrm{Ext}\, A)^c$$

$$\overline{A^c}^c = \mathrm{Ext}(A^c) \Rightarrow \overline{A^c} = \mathrm{Ext}(A^c)^c$$

So: $\partial A = \overline{A} \cap \overline{A^c}$

$$= \mathrm{Ext}(A)^c \cap \mathrm{Ext}(A^c)^c$$

$$= \left[\mathrm{Ext}(A) \cup \mathrm{Ext}(A^c)\right]^c$$

$$= X \setminus \left(\mathrm{Ext}(A) \cup \mathrm{Ext}(A^c)\right)$$

Moreover: $x \in \mathrm{Ext}(A^c) \Leftrightarrow \exists\, U \in \mathcal{V}(x),\ U \subseteq (A^c)^c$

$$\Leftrightarrow \exists\, U \in \mathcal{V}(x)\ \ U \subseteq A$$

$$\Leftrightarrow x \in \mathrm{Int}(A).$$

So $\partial A = X \setminus \left(\mathrm{Ext}(A) \cup \mathrm{Int}(A)\right).$