Is M continuous with respect to L^2 distance?

No.

$$f_n(x) = x^n$$

$$f_n \longrightarrow 0 \qquad\qquad f_n \longrightarrow 0$$

$$M(f_n) = 1 \qquad\qquad M(f_n) \not\longrightarrow M(0)$$

$$M(0) = 0$$

Exercise: M is continuous with respect to L^∞ distance

Note however, that the metric is showing up only indirectly in the notion of continuity via the notion of convergent sequences.

It could be that two different metrics determine the same convergent sequences. (in which case it is easy to see they determine the same convergent functions).

Trivially, one could simply scale distance

$$d_1 = d \qquad d_2 = 5d$$

Its easy to see $x_n \xrightarrow[d_1]{} x \iff x_n \xrightarrow[d_2]{} x$.

More interesting: our friends $\ell_1, \ell_2, \ell_\infty$ distance

Lemma: Suppose d and d' are metrics on X and there exists a constant C such that

$$d(x,y) \leq C\, d'(x,y) \qquad \forall\, x,y \in X$$

Then if $x_n \xrightarrow[d']{} x$ then $x_n \xrightarrow[d]{} x$.

Def: Two metrics d, d' are <u>equivalent</u> if $\exists\, c, C$

$$c\, d'(x,y) \leq d(x,y) \leq C\, d'(x,y) \qquad \forall\, x,y \in X.$$

Equivalent metrics determine the same convergent sequences (and hence the same convergent functions)

I claim d_1, d_2 and d_∞ are equivalent.

Indeed:

Exercise: $d_\infty \leq d_2$, $d_2 \leq \sqrt{2}\, d_\infty$

$d_\infty \leq d_1$ $d_1 \leq 2\, d_\infty$

So these metrics all determine the same convergent sequences (and hence the same convergent functions)

Convergence + continuity are more primitive notions than the metric itself.

Last class: a) If two metrics determine the same convergent seqs, they determine the same cts functions

b) Equivalent metrics determine the same convergent sequences. (and hence same cts functions)

Def: Let (X,d) be a metric space. Let $x \in X$, and let $r > 0$.

The (open) ball about x of radius r is

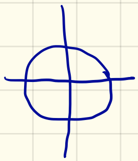
$$B_r(x) = \{ y \in X,\ d(x,y) < r \}$$

e.g. 1) $(\mathbb{R}, |\cdot|)$

$$B_1(0) = (-1,1)$$

2) $(\mathbb{R}, d_2)$

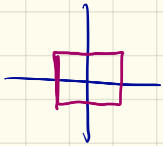
$$B_1(0) = \{ (x,y) : (x^2+y^2)^{1/2} < 1 \}$$



3) $(\mathbb{R}, d_\infty)$

$$B_1(0) = \{ (x,y) : \max(|x-0|, |y-0|) < 1 \}$$

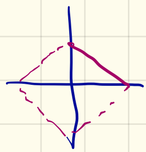
$$= \{ (x,y) : \max(|x|, |y|) < 1 \}.$$



4) $(\mathbb{R}, d_1)$

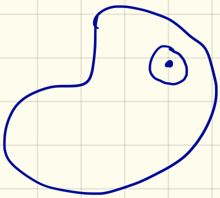
$$B_1(0) = \{ (x,y) : |x| + |y| < 1 \}.$$

in first quadrant: $x+y<1$ $\quad$ $x+y=1$ $\quad$ $y = 1-x$



Def: Let (X,d) be a metric space.

A set $A \subseteq X$ is <u>open</u> if $\forall a \in A$ there exists $r>0$ such that $B_r(a) \subseteq A$.

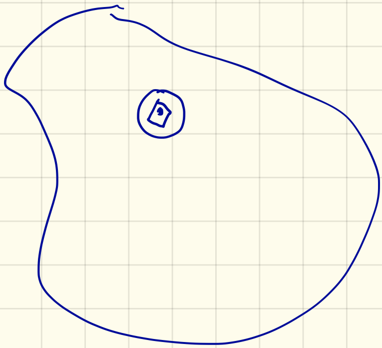


e.g. $\{0\} \subseteq \mathbb{R}$ is not open, nor is $(0,1]$.

Exercise: A ball in a metric space is open.

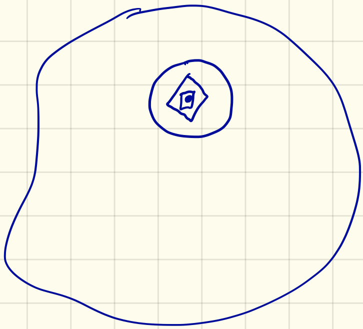
(Use Δ ineq!)

Exercise: Suppose d and d' are equivalent. Then $U \subseteq X$ is open w.r.t. d iff it is open with respect to d'.



$\boxed{\text{Equivalent metrics determine same open sets.}}$

$\ell_1, \ell_2, \ell_\infty$:



Def: A set $A \subseteq X$ is closed if A^c ($= X \setminus A$) is open.

The complement of a ball is closed.

Exercise: Show $(-\infty, -1)$ is open and $(1, \infty)$ is as well.

Show that the union of two open sets is open.

Conclude $[-1, 1]$ is closed.

Every metric d on X determines the collection τ of open sets on X, and the set $\mathcal{F}$ of closed subsets of X.

Clearly τ and $\mathcal{F}$ are closely related (if you know τ ~~or~~ $\mathcal{F}$, then you know the other.)

In particular, equivalent metrics determine the same closed sets.

We have seen: if two metrics are equivalent

1) They determine same convergent sequences

2) continuous functions

3) open sets

4) closed sets.

This suggests there is a more fundamental notion than distance. A good first guess would be to study equivalence classes of metrics.

$\because$

Motivating Theorem

Suppose d_1 and d_2 are two metrics on X.
Then TFAE

1) For all sequences $\{x_n\}$, if $x_n \xrightarrow[d_2]{} x$, then $x_n \xrightarrow[d_1]{} x$,

2) For all functions $f: X \to \mathbb{R}$, if f is cts w.r.t. d_1 then f is cts w.r.t d_2

3) For all $U \subseteq X$, if U is open w.r.t. d_1 then U is open w.r.t. d_2

4) For all $V \subseteq X$, if V is closed w.r.t d_1 then U is closed w.r.t d_2.