Name:

1. Define what it means for the vectors x_1, x_2 and x_3 to be linearly dependent. Your definition should involve the phrase "there exist".

There exist numbers β_1, β_2 and β_3 that are not all zero but such that

$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0$$

2. Give an example of three **nonzero** vectors x_1, x_2 and x_3 in $\mathbb{R}^3$ such that the set of three vectors is linearly dependent. **Justify** your claim. For full credit, each vector x_i must be different from the other two.

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$2x_1 - x_2 + 0x_3 = 0$$

$\beta_1 = 2$, $\beta_2 = -1$, $\beta_3 = 0$ (not all zero)

3. Consider the vectors

$$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Notice that

$$2x_1 + x_2 - x_3 = 0$$

and that

$$x_1 + 2x_2 + x_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}. \tag{1}$$

No work for you so far! But now: find a solution of

$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

that is **not** the solution $\beta_1 = 1, \beta_2 = 2, \beta_3 = 1$ from equation (1).

For any number α,

$$\left(x_1 + 2x_2 + x_3\right) + \alpha\left(2x_1 + x_2 - x_3\right) = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

e.g. $\alpha = 3$ $\quad 7x_1 + 5x_2 - 2x_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\beta_1 = 7, \ \beta_2 = 5, \ \beta_3 = -2$

In fact

$$(\beta_1, \beta_2, \beta_3) = (1, 2, 1) + \alpha(2, 1, -1)$$

works for any choice of α.