1. Consider the vectors $v_1 = (1, 2, 1, -2)$ and $v_2 = (1, 2, 3, 4)$. Let V be the collection of all linear combinations of v_1 and v_2. This is a two dimensional plane in $\mathbb{R}^4$. Let W be the collection of all vectors that are perpendicular to all the vectors in V. This is known as the orthogonal complement of V and is sometimes written $W = V^{\perp}$.

1. Show that if $w \in W$ then $v_1^T w = 0$ and $v_2^T w = 0$. Yes, this is an easy question.
2. Show that if w is a vector satisfying the two conditions $v_1^T w = 0$ and $v_2^T w = 0$ then in fact $w \in W$. Hint: an arbitrary element in V has the form $v = c_1 v_1 + c_2 v_2$. Now take some dot products.
3. The set of vectors w satisfying $v_1^T w = 0$ and $v_2^T w = 0$ is the nullspace of a specific matrix. What is the matrix?
4. Determine the nullspace of this matrix to determine all the vectors in W.

2. A 4×4 matrix has $\det(A) = 1/3$. Find $\det(2A)$, $\det(-A)$, $\det(A^2)$ and $\det(A^{-1})$.

3. Find two 2×2 matrices A and B with $\det(A) = 1$ and $\det(B) = 1$ but $\det(A + B) = 0$. So there is no rule $\det(A + B) = \det(A) + \det(B)$.

4. Compute the determinant of

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}$$

by reducing to an upper triangular matrix.

5. Compute the determinant of

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 10 \\ 5 & 6 & 7 \end{bmatrix}$$

by reducing to an upper triangular matrix. Note that you will need to keep track of row interchanges.

6. Find the determinant of

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}.$$

Then explain your result.

7. Compute the determinant of

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

by expansion along the third row, i.e. the long formula with six terms coming from three 2×2 determinants.

8. Compute the determinant of

$$A = \begin{bmatrix} 5 & 1 & -1 & 2 & 1 \\ 3 & 0 & 0 & 0 & 3 \\ 2 & 3 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 6 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

You'll want to choose your expansion row wisely...

9. The matrix E_k is a $k \times k$ matrix that is all zeros, except for the entries on or adjacent to the main diagonal, where the values are 1s. For example,

$$E_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

1. Compute $\det(E_1)$, $\det(E_2)$ and $\det(E_3)$. Note that

$$E_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

2. Show that $\det(E_5) = \det(E_4) - \det(E_3)$ by expanding on the first row. You don't need to compute the number, just show that the formula holds.

3. This formula holds generally: $\det(E_{k+1}) = \det(E_k) - \det(E_{k-1})$. Using this fact, compute E_k for $k = 3, 4, 5, 6, 7, 8$.

4. There's a pattern here! Use the pattern to compute E_{100}.

10. Compute the determinant of the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

two different ways. First, reduce to upper triangular. Second, use expansion along the first row.