Things you can represent:

all the above examples

assets $\longrightarrow$ revenue per month for months

signal $\longrightarrow$ discrete derivative

coeffs $\longrightarrow$ polynomial evaluated at points

vector $\longrightarrow$ itself (I)

Heres some more

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}$$

permutation matrix

row i $\longrightarrow$ row j

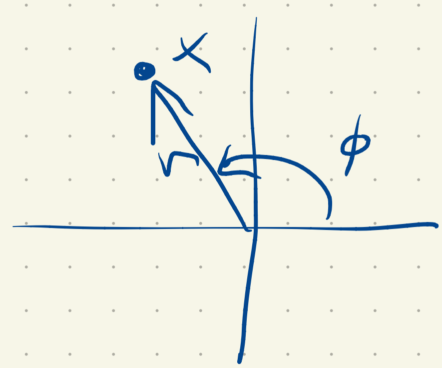
if column i is e_j

Recall: $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\sin(A+B) = \sin A \cos B + \cos B \sin A$$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$x = \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix}$$

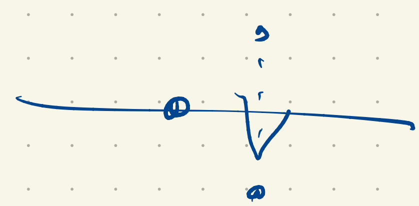


$$R_\theta x = r\begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{bmatrix} = r\begin{bmatrix} \cos(\phi+\theta) \\ \sin(\phi+\theta) \end{bmatrix}$$

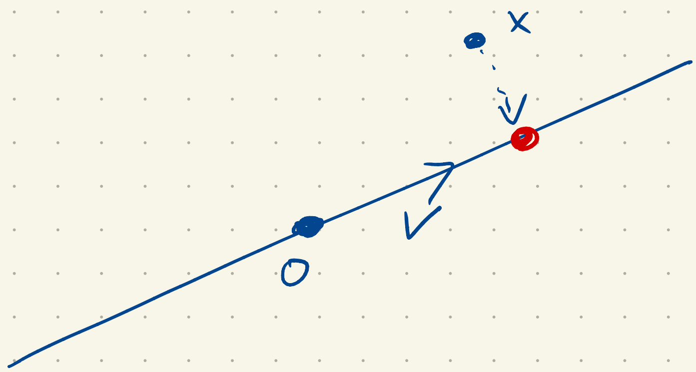
Rotation of the plane by θ

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

reflect about x-axis.



Can rep all reflectors like this.



$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

ortho projection onto x-axis

$$x \longrightarrow \left(\frac{x^T v}{\|v\|^2} \right) v$$

$$v^T \left(x - \frac{x^T v}{\|v\|^2} v \right) = 0 \checkmark$$

$$\frac{1}{\|v\|^2} \begin{bmatrix} v_1 v_1 & v_1 v_2 \\ v_2 v_1 & v_2 v_2 \end{bmatrix}$$

$$A_{ij} = \frac{v_i v_j}{\|v\|^2}$$

$$\sum A_{ij} x_j = v_i \frac{\sum v_j x_j}{\|v\|^2} = \frac{x^T v}{\|v\|^2} v_i$$

Downsampling:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix}$$

Smoothing:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} (x_1+x_2)/2 \\ (x_1+x_2+x_3)/3 \\ (x_2+x_3+x_4)/3 \\ (x_3+x_4+x_5)/3 \\ (x_4+x_5)/2 \end{bmatrix}$$

closely related: convolution:

$a = (a_1, a_2, a_3)$

$$\begin{bmatrix} a_1 & 0 & 0 \\ a_2 & a_1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_3 & a_2 \\ 0 & 0 & a_3 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ 0 \\ 0 \end{matrix} = \begin{bmatrix} a_1 b_1 \\ a_2 b_1 + a_1 b_2 \\ a_3 b_1 + a_2 b_2 + a_1 b_3 \\ a_3 b_2 + a_2 b_3 \\ a_3 b_3 \end{bmatrix}$$

$$(a_3x^2 + a_2x + a_1)(b_3x^2 + b_2x + b_1)$$

$$= a_3b_3x^4 + (a_3b_2 + a_2b_3)x^3 + (a_3b_1 + a_2b_2 + a_1b_3)x^2$$
$$+ (a_2b_1 + a_1b_2)x$$
$$+ a_1b_1$$

If $a_1 = a_2 = a_3 = \frac{1}{3}$ this is nearly smoothing.

a a vector, b a vector,

$$\overbrace{(b_1, b_2, b_3, \ldots, b_n)}^{\text{data from here}}$$

$$\underbrace{(a_3, a_2, a_1)}_{\text{weights from here}} \longrightarrow \quad a * b$$

or $b * a$!

If $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ then $a * b \in \mathbb{R}^{n+m-1}$

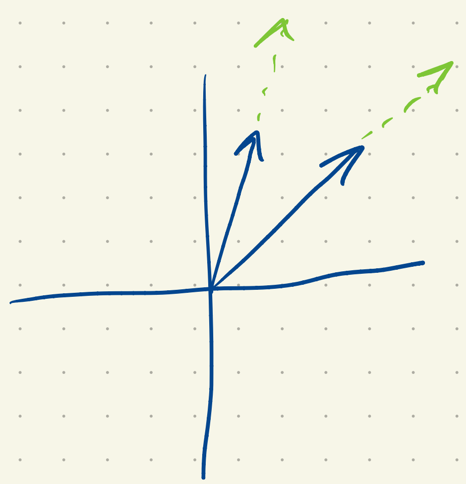
Linear Functions

A map $f: \mathbb{R}^n \to \mathbb{R}^m$ is linear if

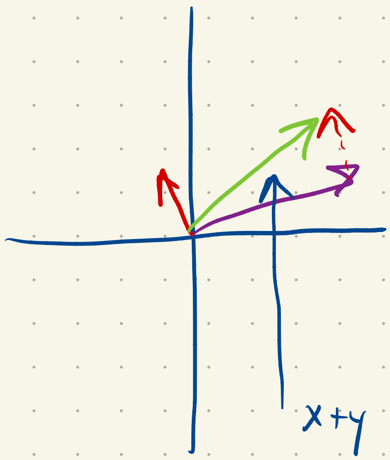
$$f(x+y) = f(x) + f(y) \qquad \forall x, y \in \mathbb{R}^n$$
$$f(cx) = c f(x) \qquad c \in \mathbb{R}$$

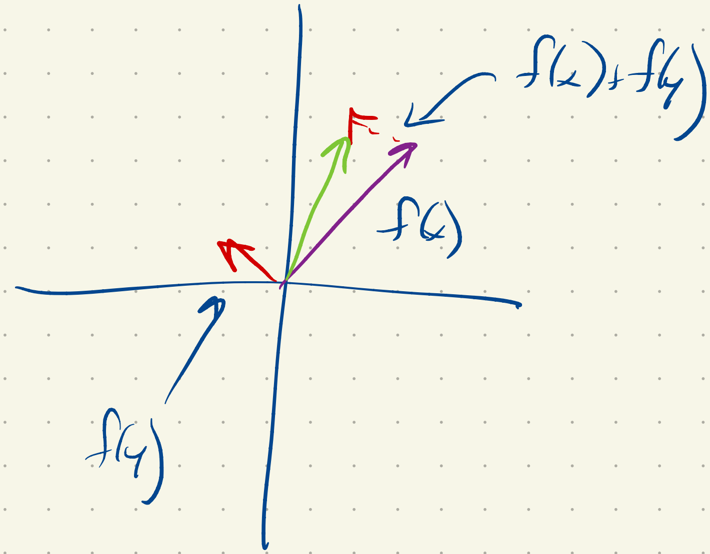
$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \qquad \forall \alpha, \beta \in \mathbb{R},\ x, y \in \mathbb{R}^n$$

Is rotation linear?



$f(cx) = c f(x)$ ✓





Yep!

$$R_\theta(x+y) = R_\theta x + R_\theta y$$

$$R_\theta(cx) = cR_\theta x$$

Given an $m \times n$ matrix A,

$$f_A(x) = Ax \qquad f_A : \mathbb{R}^n \to \mathbb{R}^m$$

f_A is linear same argument as above

In fact, if f is linear, then it comes from matrix multiplication

$$\underbrace{e_1, \dots, e_n}$$

$\hookrightarrow$ If you know $f(e_j)$ then you know everything f does.

$$x = x_1 e_1 + \dots + x_n e_n$$

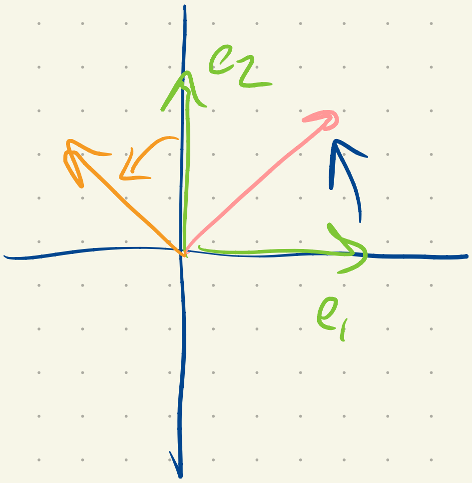
$$f(x) = x_1 f(e_1) + \dots + x_n f(e_n)$$

$$= \underbrace{\begin{bmatrix} f(e_1) & \cdots & f(e_n) \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$f = f_A.$$

e.g. $R_{\frac{\pi}{4}}$ $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$



$$R_{\frac{\pi}{4}}(e_1) = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$R_{\frac{\pi}{4}}(e_2) = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{array}{cc} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ \uparrow \quad\quad \uparrow \\ \text{where does } e_1 \text{ go} \quad \text{where does } e_2 \text{ go?} \end{array}$$