Last class:

We discussed the dimension independence inequality:

If you have $n+1$ or more vectors in $\mathbb{R}^n$ then they are linearly dependent.

I hope to video the proof of this for you

But you can certainly have n linearly independent vectors in $\mathbb{R}^n$. E.g. in $\mathbb{R}^3$,

$e_1 \quad e_2 \quad e_3$ work

$$\beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 = 0 \Rightarrow \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = 0 \Rightarrow \text{all zero.}$$

Same argument works in all dimensions.

Def: A <u>basis</u> for $\mathbb{R}^n$ is a collection of n linearly independent vectors in $\mathbb{R}^n$.

Why care?

Claim: Given a basis $x_1, \ldots, x_n$ for $\mathbb{R}^n$ and $y \in \mathbb{R}^n$ there are unique numbers $\beta_1, \ldots, \beta_n$ with

$$\beta_1 x_1 + \cdots + \beta_n x_n = y.$$

This is about solving equations.

Uniqueness is a consequence of linear independence

Why existence

$x_1, \ldots, x_n, y$ are linearly dependent.

$\alpha_1 x_1 + \cdots + \alpha_n x_n + \alpha_{n+1} y = 0$ not all $\alpha_i = 0$

$\alpha_{n+1} \neq 0$. otherwise $x_1, \ldots, x_n$ lin dep

$$y = -\frac{\alpha_1}{\alpha_{n+1}} x_1 + \cdots - \frac{\alpha_n}{\alpha_{n+1}} x_n.$$

Great.

Now given a basis for $\mathbb{R}^n$ and y how can you find β's?

$x_1, \ldots, x_n$

e.g. $x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $x_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

These are a basis for $\mathbb{R}^2$. Just need to show lin ind.

$$\beta_1 x_1 + \beta_2 x_2 = 0 \Rightarrow \begin{aligned} 2\beta_1 + \beta_2 &= 0 \\ \beta_1 + 3\beta_2 &= 0 \end{aligned} \Rightarrow \beta_1 = -3\beta_2$$

$$\Rightarrow -5\beta_2 = 0$$

$$\Rightarrow \beta_1 = \beta_2 = 0$$

So: There exist c_1, c_2

$$c_1 x_1 + c_2 x_2 = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

How to find? In general, this is really hard.

$$\begin{aligned} 2c_1 + c_2 &= 5 \\ c_1 + 3c_2 &= 7 \end{aligned}$$

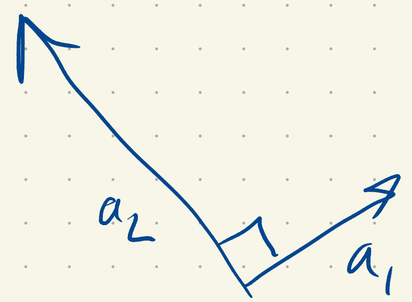
(and bigger in $\mathbb{R}^{100}$)

You are solving. It's hard. But! Some bases are better than others!

Orthonormality.

A collection of vectors $a_1, \ldots, a_k$ is **orthogonal** if $a_i^T a_j = 0$ $i \neq j$.

They are mutually perpendicular.



The collection is orthonormal if in addition

$$\|a_j\| = 1 \quad 1 \leq j \leq k.$$

I.e.

$$a_i^T a_j = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

E.g. $a_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$ $a_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ $a_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

An orthonormal collection $a_1, \dots, a_k$ is

<u>always</u> linearly independent.

Suppose $\beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_n a_n = 0$.

$$a_1^T(\qquad\qquad) = \beta_1 \|a_1\|^2 + \beta_2 \cdot 0 + \cdots + \beta_n \cdot 0 = 0$$

$$\Rightarrow \beta_1 = 0.$$

Take dot product with a_j and conclude $\beta_j = 0$.

Suppose $a_1, \dots, a_n$ are orthonormal in $\mathbb{R}^n$.

They are linearly independent. Hence

given $x \in \mathbb{R}^n$ we can write

$$x = c_1 a_1 + \cdots + c_n a_n.$$

for some c_j's. How can we compute?

$$a_j^T x = c_1 \, a_j^T a_1 + \dots + c_j a_j^T a_j + \dots + c_n a_j^T a_n$$

$$= 0 + \dots + c_j + \dots + 0.$$

$$= c_j$$

Whoa!

e.g. $x = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$$a_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \quad a_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad a_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$a_1^T x = \frac{3}{\sqrt{2}} \qquad a_2^T x = -\frac{1}{\sqrt{2}} \qquad a_3^T x = 2$$

Claim $x = \frac{3}{\sqrt{2}} a_1 - \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + 2 a_3$

$$= \begin{bmatrix} 3/2 \\ 0 \\ -3/2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \\ -1/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$